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Heat Kernel Analysis and Cameron–Martin Subgroup for Infinite Dimensional Groups

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The heat kernel measure μ_t is constructed on $GL(H)$, the group of invertible

Martin subgroup G_{CM} is defined and its properties are discussed. In particular, there is an isometry from the $L^2_{\mu_t}$-closure of holomorphic polynomials into a space $\mathscr{H}^t(G_{\mathrm{CM}})$ of functions holomorphic on G_{CM}. This means that any element from this $L^2_{\mu_t}$-closure of holomorphic polynomials has a version holomorphic on G_{CM}. In addition, there is an isometry from $\mathscr{H}^t(G_{\mathrm{CM}})$ into a Hilbert space associated with the tensor algebra over $\mathfrak{g}$. The latter isometry is an infinite dimensional analog of the Taylor expansion. As examples we discuss a complex orthogonal group and a complex symplectic group. © 2000 Academic Press

Key Words: heat kernel measure; holomorphic function; infinite dimensional group; infinite dimensional Lie algebra; stochastic differential equation.

Contents

1. INTRODUCTION

Our goal is to study Hilbert spaces of holomorphic functions on a group associated with an infinite dimensional Lie algebra $\mathfrak{g}$ which is itself

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equipped with a Hermitian inner product $(\cdot,\cdot)$ and corresponding norm $|\cdot|$. We assume that $\mathfrak{g}$ is a Lie subalgebra of $B(H)$, the space of bounded linear operators on a complex separable Hilbert space H. The group under consideration is a Lie subgroup of $GL(H)$, the group of invertible elements of $B(H)$. Note that $B(H)$ is the natural (infinite dimensional) Lie algebra of $GL(H)$ with the commutator as a Lie bracket.

One of the main ingredients in this work is the construction of the heat kernel measure on $GL(H)$ which is determined by $\mathfrak{g}$ and the norm $|\cdot|$. In some cases it is possible to show that the heat kernel measure is supported in a subgroup of $GL(H)$ (see Section 9). The construction of the heat kernel measure requires the use of stochastic differential equations in a Hilbert space. We will assume that $\mathfrak{g}$ is a subspace of the Hilbert–Schmidt operators on H.

It is well known that the Cameron–Martin subspace plays an important role for an abstract Wiener space. Analogously, we define the Cameron–Martin subgroup, G_{CM}, and discuss its properties. One of these properties is that functions in the L^2-closure of holomorphic polynomials have holomorphic versions on G_{CM}. Following [15, 16] we call these versions skeletons. The map taking an L^2-function to its skeleton is an isometry to $\mathscr{H}^t(G_{\mathrm{CM}})$, a space of functions holomorphic on G_{CM} with a direct limit-type norm derived from finite dimensional approximations to G_{CM}. We also show that the Taylor map, from holomorphic functions on G_{CM} into a dual of the universal enveloping algebra of $\mathfrak{g}$, is isometric on $\mathscr{H}^t(G_{\mathrm{CM}})$. This isometry is a noncommutative version of one of the isomorphisms between different representations of a bosonic Fock space.

An outline of the history of the subject has been given in [7, 8]. We should mention here works by Sugita [15, 16] for an abstract complex Wiener space, in particular, his results on skeletons of L^2-functions on the Cameron–Martin subspace.

2. NOTATION AND MAIN RESULTS

To describe results of this paper in more detail we need to use finite dimensional approximations to $\mathfrak{g}$. Let $G_1 \subseteq G_2 \subseteq \cdots \subseteq G_n \subseteq \cdots \subseteq B(H)$ be a sequence of complex connected (finite dimensional) Lie subgroups of $GL(H)$. We assume that their Lie algebras $\mathfrak{g}_n \subseteq B(H)$ are equipped with consistent Hermitian inner products, that is, $(\cdot,\ \cdot)_{n+m}|_{\mathfrak{g}_n} = (\cdot,\ \cdot)_n$, where $(\cdot,\ \cdot)_n$ is the inner product on $\mathfrak{g}_n$. The corresponding norm is denoted by $|\cdot|_n$.

Let $\mathfrak{g} = \bigcup_{n=1}^{\infty} \mathfrak{g}_n$ with the Hermitian inner product $(\xi, \eta) = (\xi, \eta)_n$ for any $\xi, \eta \in \mathfrak{g}_n$. We assume that $|x| \geqslant \|x\|$ for any $x \in \mathfrak{g}$, where $\|\cdot\|$ is the operator norm. Denote by $\mathfrak{g}_\infty$ the closure of $\mathfrak{g}$ in the norm $|\cdot|$, that is, all elements of finite norm. We also assume that the closure coincides with the completion of $\mathfrak{g}$ with respect to the norm $|\cdot|$. Note that $\mathfrak{g}_\infty$ is a subset of $B(H)$.

Let d denote the Riemannian metric $d(y, z) = \inf_h \{ \int_0^1 |h^{-1}\dot{h}| \, ds \}$, where h: $[0, 1] \to GL(H)$ is a piecewise differentiable path, $h(0) = y$, $h(1) = z$, $\dot{h} = dh/ds$, $h' = h^{-1}\dot{h} \in \mathfrak{g}_\infty$. Let G_∞ be the closure of $\bigcup_{n=1}^\infty G_n$ in the Riemannian metric $d_\infty = \inf_n d_n$, where d_n is the Riemannian metric on G_n. Again we assume that the closure coincides with the completion. By $\mathscr{H}^t(G_\infty)$ we denote a space of holomorphic functions on G_∞ with a certain direct limit-type norm $\|\cdot\|_{t,\infty}$. The precise definition is given in Section 4.

Let $(1-D)_X^{-1} f = \sum_{k=0}^\infty (D^k f)(X)$ be the series of all derivatives of a function f on G_∞. Then the Taylor map $(1-D)_I^{-1}$ is an isometry from $\mathscr{H}^t(G_\infty)$ into a subspace, J_t^0, of the dual of the tensor algebra of $\mathfrak{g}_\infty$ equipped with the norm

$$|\alpha|_t^2 = \sum_{k=0}^\infty \frac{t^k}{k!} |\alpha_k|^2, \qquad \alpha = \sum_{k=0}^\infty \alpha_k, \quad \alpha_k \in (\mathfrak{g}^{\otimes k})^*, \quad k = 0, 1, 2, \ldots, t > 0.$$

See Notation 3.2 and Notation 4.2 for more detailed definitions. The following theorem will be proved in Section 4.

Theorem. *$\mathscr{H}^t(G_\infty)$ is a Hilbert space and $(1-D)_I^{-1}$ is an isometry from $\mathscr{H}^t(G_\infty)$ into J_t^0.*

Moreover, in case the G_n are simply connected Theorem 4.5 gives the image of $(1-D)_I^{-1}$, which can be informally described as a completion of the universal enveloping algebra.

The heat kernel measure is constructed in Section 5 using stochastic differential equations on Hilbert spaces, in this case on the space of Hilbert–Schmidt operators. Denote by HS the space of Hilbert–Schmidt operators on H with the Hilbert–Schmidt (Hermitian) inner product $(\cdot, \cdot)_{HS}$ and corresponding real inner product $\langle \cdot, \cdot \rangle_{HS} = \mathrm{Re}(\cdot, \cdot)_{HS}$. In most of the results of this paper we assume that $G_n \subset I + HS$, $\mathfrak{g}_n \subset HS$. The following is a summary of results contained in Theorem 5.1 and Theorem 5.4.

Theorem. *Let W_t be the Wiener process in HS with covariance determined by the norm on $\mathfrak{g}_\infty$. Then the stochastic differential equation*

$$dX_t = X_t \, dW_t,$$

$$X_0 = I$$

has a unique solution in $(I + HS) \cap GL(H)$.

The transition probability of the process X_t determines the fundamental solution of the heat equation with the informal Laplacian

$$(\Delta v)(X) = \tfrac{1}{2} \sum_{n=1}^{\infty} (\tilde{\xi}_n \tilde{\xi}_n v)(X),$$

where $\{\xi_n\}_{n=1}^{\infty}$ is an orthonormal basis of $\mathfrak{g}_\infty$ as a real space with the real inner product $\langle \xi, \eta \rangle = \mathrm{Re}(\xi, \eta)$ and $(\tilde{\xi}_n v)(X) = d/dt|_{t=0}\, v(Xe^{t\xi_n})$ for a function v: $I + HS \to \mathbb{R}$. The corresponding measure is called the heat kernel measure and the space of functions square integrable, with respect to this measure, is denoted by $L^2(I+HS, \mu_t)$ and the norm $\|\cdot\|_t$.

In addition to G_∞ we consider the Cameron–Martin subgroup.

Definition 2.1. $G_{\mathrm{CM}} = \{x \in B(H) : d(x, I) < \infty\}$ is called *the Cameron–Martin subgroup.*

Proposition 7.1 proves that G_{CM} is a group. Note that $G_\infty \subset G_{\mathrm{CM}}$ and the following theorem shows that under a condition on the Lie bracket they are actually equal. The next theorem will be proved in Section 7.

Theorem. *If* $|[x, y]| \leqslant C\, |x|\, |y|$ *for all* $x, y \in \mathfrak{g}_\infty$, *then*

1. $G_{\mathrm{CM}} = G_\infty$.
2. *The exponential map is a diffeomorphism from a neighborhood of* 0 *onto a neighborhood of* I *in* G_{CM}.

Note that under the condition of this theorem $\mathfrak{g}_\infty$ is a Lie algebra.

Naturally defined holomorphic polynomials on $I + HS$ play an important role in several results. One of them is that there is a natural isometry from the space of holomorphic polynomials $\mathcal{HP}$ to $\mathcal{H}^t(G_\infty)$. To prove this we use approximations to the process $Y_t + I$ discussed in Section 6. In addition, this isometry defines holomorphic skeletons on G_{CM} for the elements of the closure of $\mathcal{HP}$ in $L^2(I+HS, \mu_t)$. This closure is denoted by $\mathcal{H}L^2(I+HS, \mu_t)$. The following results are contained in Theorem 8.7 and Theorem 8.5.

Theorem. 1. $\mathcal{HP} \subset L^2(I+HS, \mu_t)$.

2. *The identity* $\|f\|_{t,\infty} = \|f\|_t$ *for any* $f \in \mathcal{HP}$ *extends to an isometry* $\mathfrak{I}_{G_\infty}$ *from* $\mathcal{H}L^2(I+HS, \mu_t)$ *into* $\mathcal{H}^t(G_\infty)$.

3. *Suppose* $p_n \xrightarrow[n \to \infty]{L^2(I+HS, \mu_t)} f$, $p_n \in \mathcal{HP}$. *Then there is a holomorphic function* $\tilde{f}$, *a skeleton of* f, *on* G_{CM} *such that* $p_n(x) \to \tilde{f}(x)$ *for any* $x \in G_{\mathrm{CM}}$.

We will denote the skeleton map by $\mathfrak{I}_{G_{\mathrm{CM}}}$. Note that elements of $\mathcal{H}L^2(I+HS, \mu_t)$ are defined up to a set of μ_t-measure zero. Still the map

$\mathfrak{I}_{G_{\mathrm{CM}}}$ gives a holomorphic version on G_{CM} of any element from $\mathscr{H}L^2(I+HS, \mu_t)$, even though G_{CM} itself might be of μ_t-measure zero. In addition, $\mathfrak{I}_{G_{\mathrm{CM}}} f|_{G_\infty}$ is actually the isometry $\mathfrak{I}_{G_\infty}$ from part 2 of the last theorem. This means that for holomorphic polynomials the skeleton map is the restriction map (to G_{CM}).

Finally, Section 9 provides several examples to this abstract setting. One of the examples is the Hilbert–Schmidt complex orthogonal group which has been discussed in [7]. In addition we consider the Hilbert–Schmidt complex symplectic group and a group of infinite diagonal matrices.

For some $\mathfrak{g}$ and natural norms $|\cdot|$ on it, the Taylor map is an isometry between trivial spaces (see Section 9). But we show that for the natural condition on the norm $|\cdot|$ considered in this paper, there are non-constant functions in $\mathscr{H}^t(G_\infty)$. Namely, this space contains all holomorphic polynomials. Indeed, Theorem 8.7 proves that the holomorphic polynomials are square integrable with respect to the heat kernel measure. Then in the same theorem we show that the L^2-norm of a polynomial is equal to the $\mathscr{H}^t(G_\infty)$-norm.

The following commutative diagram illustrates all the isometries described in this paper.

$$\begin{tikzcd}
 & \mathscr{H}L^2(I+HS, \mu_t) \arrow[dl, "\substack{\text{skeleton map}\\ \mathfrak{I}_{G_{\mathrm{CM}}}}"'] \arrow[dr, "\mathfrak{I}_{G_\infty}"] & & \\
\mathscr{H}^t(G_{\mathrm{CM}}) \arrow[rr, "\text{restriction map}"] & & \mathscr{H}^t(G_\infty) \arrow[r, "\text{Taylor map}"] & J_t^0
\end{tikzcd}$$

3. ESTIMATES OF DERIVATIVES OF HOLOMORPHIC FUNCTIONS

Let μ_t^n be the heat kernel measure on G_n; let $\mathscr{H}L^2(G_n, \mu_t^n)$ be the space of holomorphic functions square integrable with respect to μ_t^n.

Notation 3.1. $\|f\|_{L^2(G_n, \mu_t^n)} = \|f\|_{t,n}$.

Notation 3.2. Suppose f is a function from G_{CM} to $\mathbb{C}$. Let $(Df)(X)$ denote the unique element of $\mathfrak{g}_\infty^*$ (as a complex space) such that

$$(Df)(X)(\xi) = (\tilde{\xi} f)(X) = \left.\frac{d}{dt}\right|_{t=0} f(Xe^{t\xi}), \qquad \xi \in \mathfrak{g}_\infty, \quad X \in G_\infty(G_{\mathrm{CM}}),$$

if the derivative exists. Similarly $(D^k f)(X)$ denotes the unique element of $(\mathfrak{g}_\infty^{\otimes k})^*$ such that

$$(D^k f)(X)(\beta) = (\tilde{\beta} f)(X), \qquad \beta \in \mathfrak{g}_\infty^{\otimes k}, \quad X \in G_{\mathrm{CM}}$$

and $(D_n^k f)(X)$ denotes a unique element of $(\mathfrak{g}_n^*)^{\otimes k} = (\mathfrak{g}_n^{\otimes k})^*$ such that

$$(D_n^k f)(X)(\beta) = (\tilde{\beta} f)(X), \qquad \beta \in \mathfrak{g}_n^{\otimes k}, \quad X \in G_n.$$

We will use the following notation:

$$(1-D)_X^{-1} f = \sum_{k=0}^{\infty} (D^k f)(X) \qquad \text{and} \qquad (1-D_n)_X^{-1} f = \sum_{k=0}^{\infty} (D_n^k f)(X).$$

The following estimate was proved by Driver and Gross in [5] for $f \in \mathscr{H}L^2(G_n, \mu_t^n)$,

$$|(\tilde{\beta} f)(g)|^2_{(\mathfrak{g}_n^{\otimes k})^*} \leqslant \|f\|^2_{t,n} \frac{k!\, |\beta|_n^2}{r^k} e^{|g|_n^2/s},$$

$$\text{for} \quad g \in G_n, \quad r > 0, \quad s + r \leqslant t, \quad \beta \in \mathfrak{g}_n^{\otimes k},$$

where $|(\tilde{\beta} f)(g)|_{(\mathfrak{g}_n^{\otimes k})^*}$ is a $((T_g G_n)^*)^{\otimes k}$-norm (which can be identified with $(\mathfrak{g}_n^*)^{\otimes k}$) and $|g|_n = d_n(I, g)$. We will need a slight modification of this estimate. Taking supremum over all $\beta \in \mathfrak{g}_n^{\otimes k}$, $|\beta|_n = 1$, we get

$$|(D_n^k f)(g)|^2 \leqslant \|f\|^2_{t,n} \frac{k!}{r^k} e^{|g|_n^2/s}, \tag{3.1}$$

where D_n^k is defined for G_n and $\mathfrak{g}_n$ by Notation 3.2. Note that if $\|f\|_{t,n}$ are uniformly bounded, then (3.1) gives a uniform bound, i.e., a bound independent of n. The following estimates can be proved for D_n^k:

Lemma 3.3. *Let $r > 0$, $q + r \leqslant t$, X, $Y \in G_n$, $f \in \mathscr{H}L^2(G_n, \mu_t^n)$. Then*

$$|(D_n^k f)(X) - (D_n^k f)(Y)|_{(\mathfrak{g}_n^{\otimes k})^*} \leqslant \|f\|_{t,n} K_{k+1,n}\, d_n(X, Y),$$

where $K_{k,n} = K_{k,n}(X, Y) = (k!/r^k)^{1/2} e^{|X|_n^2 + d_n(X,Y)^2/q}$.

Proof. Take $h\colon [0,1] \to G_n$ such that $h(0) = X$, $h(1) = Y$. Then by (3.1)

$$\begin{aligned} &|D_n^k f(X) - D_n^k f(Y)|_{(\mathfrak{g}_n^{\otimes k})^*} \\ &\quad = \left| \int_0^1 \frac{d}{ds} (D_n^k f)(h(s))\, ds \right|_{(\mathfrak{g}_n^{\otimes k})^*} \\ &\quad \leqslant \int_0^1 \left| \frac{d}{ds} (D_n^k f)(h(s)) \right|_{(\mathfrak{g}_n^{\otimes k})^*} ds \\ &\quad \leqslant \int_0^1 |D(D_n^k f)(h(s))(h^{-1}\dot{h})|_{(\mathfrak{g}_n^{\otimes k})^*}\, ds \end{aligned}$$

$$\leqslant \int_0^1 |D(D_n^k f)(h(s))|_{(\mathfrak{g}_n^{\otimes k+1})^*} \, |h^{-1}\dot{h}|_n \, ds$$

$$\leqslant \|f\|_{t,n} \left(\frac{(k+1)!}{r^{k+1}} \sup_{u \in [0,1]} e^{|h(u)|_n^2/q} \right)^{1/2} \int_0^1 |h^{-1}\dot{h}|_n \, ds$$

$$\leqslant \|f\|_{t,n} \left(\frac{(k+1)!}{r^{k+1}} \right)^{1+2} \sup_{u \in [0,1]} e^{|h(0)|_n^2 + d_n(h(0), h(u))^2/q} \int_0^1 |h^{-1}\dot{h}|_n \, ds.$$

Taking infimum over all such h we see that

$$|D_n^k f(X) - D_n^k f(Y)|_{(\mathfrak{g}_n^{\otimes k})^*} \leqslant \|f\|_{t,n} \left(\frac{(k+1)!}{r^{k+1}} \right)^{1/2} e^{|X|_n^2 + d_n(X, Y)^2/q} \, d_n(X, Y). \quad \blacksquare$$

LEMMA 3.4. *Let* $X \in G_n$, $f, g \in \mathscr{H}L^2(G_n, \mu_t^n)$, $t > 0$. *Then*

$$|D_n^k f(X) - D_n^k g(X)|_{(\mathfrak{g}_n^{\otimes k})^*} \leqslant M_k \, \|f - g\|_{t,n},$$

where $M_k = M_k(X, t) = (k!/(t/2)^k)^{1/2} \, e^{|X|_n^2/t}$.

Proof. From (3.1) we have that for $r > 0$, $q + r \leqslant t$

$$|D_n^k f(X) - D_n^k g(X)|^2_{(\mathfrak{g}_n^{\otimes k})^*} \leqslant \|f - g\|^2_{t,n} \frac{k!}{r^k} e^{|X|_n^2/q}.$$

Now take $q = r = t/2$ to get what we claimed. $\blacksquare$

LEMMA 3.5. *Let* $X \in G_n$, $\xi \in \mathfrak{g}_n$, $f \in \mathscr{H}L^2(G_n, \mu_t^n)$. *Then there is a constant* $C = C(X, \xi, t) > 0$ *such that*

$$\left| \frac{f(Xe^{u\xi}) - f(X)}{u} - (Df)(X)(\xi) \right| \leqslant \|f\|_{t,n} \, Cu$$

for small enough $u > 0$.

Proof. Let $h(s) = Xe^{s\xi}$, $0 \leqslant s \leqslant u$,

$$\frac{f(Xe^{u\xi}) - f(X)}{u} - (Df)(X)(\xi) = \frac{1}{u} \int_0^u \frac{d}{ds} f(h(s)) \, ds - (Df)(X)(\xi)$$

$$= \frac{1}{u} \int_0^u \left(\frac{d}{ds} f(h(s)) - (Df)(X)(\xi) \right) ds$$

$$= \frac{1}{u} \int_0^u ((Df)(h(s))(\xi) - (D_X f)(\xi)) \, ds.$$

Thus by Lemma 3.3 for any $r>0$, $q+r\leqslant t$

$$\left|\frac{f(Xe^{u\xi})-f(X)}{u}-(D_X f)(\xi)\right|$$

$$\leqslant\frac{1}{u}\int_0^u |((Df)(h(s))(\xi)-(D_X f)(\xi))|\,ds$$

$$\leqslant\frac{1}{u}\int_0^u \|f\|_{t,n}\,K_{2,n}(h(s),X)\,d_n(h(s),X)\,|\xi|_n\,ds$$

$$\leqslant\frac{1}{u}\int_0^u \|f\|_{t,n}\,\frac{\sqrt{2}}{r}\,e^{(|X|_n^2+|\xi|_n^2 s^2)/q}s\,|\xi|_n^2\,ds$$

$$=\|f\|_{t,n}\,\frac{\sqrt{2}}{r}\,e^{|X|_n^2/q}\,|\xi|_n^2\,\frac{1}{u}\int_0^u e^{(|\xi|_n^2/q)\,s^2}s\,ds$$

$$=\|f\|_{t,n}\,\frac{q}{\sqrt{2}\,r}\,e^{|X|_n^2/q}\,\frac{e^{(|\xi|_n^2/q)\,u^2}-1}{u}\leqslant\|f\|_{t,n}\,Cu$$

for small u. ∎

THEOREM 3.6. *Let f be a function on $\bigcup_n G_n$. Suppose that $f|_{G_n}$ is holomorphic for any n and $\sup_n \|f\|_{t,n}<\infty$. Then f and all its derivatives have unique continuous extensions from $\bigcup_n G_n$ to G_∞.*

Proof. Take $X\in G_\infty$. We would like to define the extensions by

$$D^k f(X)=\lim_{n\to\infty} D^k f(X_n), \qquad X_n\in G_n, \quad X_n \xrightarrow[n\to\infty]{d_\infty} X.$$

Let us check that the limits exist. Assume that $l\leqslant n$. By Lemma 3.3 for $X_n,\ X_l\in G_n$

$$|D^k f(X_n)-D^k f(X_l)|_{(\mathfrak{g}_n^*)^{\otimes k}}\leqslant\|f\|_{t,n}\,K_{k+1,n}(X_n,X_l)\,d_n(X_n,X_l).$$

Thus $D^k f(X_n)$ is a Cauchy sequence and therefore the limit exists. The uniqueness of the extension is easy to verify. ∎

Remark 3.7. In general, the extensions of derivatives might not be derivatives of the extensions. They are actually the derivatives under an assumption on the Lie bracket. This will be shown in Section 7. The estimates in this section hold for the continuous extensions of D_n^k from $\bigcup_n G_n$ to G_∞.

4. ISOMETRIES

LEMMA 4.1. *Suppose $f|_{G_n} \in \mathcal{H}L^2(G_n, \mu_t^n)$ for all n. Then $\|f\|_{t,n} \leqslant \|f\|_{t,n+1}$ for any n.*

Proof. First of all, $\|f\|_{t,n}^2 = \|(1-D_n)_I^{-1} f\|_{t,n}^2$ by the Driver–Gross isomorphism (see [5]), where $\|(1-D_n)_I^{-1} f\|_{t,n}^2 = \sum_{k=0}^{\infty} |(D_n^k f)(I)|_{t,n}^2$. Take an orthonormal basis $\{\eta_l\}_{l=1}^{d_{n+1}}$ of the complex inner product space $\mathfrak{g}_{n+1}$ such that $\{\eta_l\}_{l=1}^{d_n}$ is an orthonormal basis of $\mathfrak{g}_n$. Then

$$
\begin{aligned}
|(D_{n+1}^k f)(I)|_{t,n+1}^2 &= \sum_{1 \leqslant i_m \leqslant d_{n+1}} |\tilde{\eta}_{i_1} \cdots \tilde{\eta}_{i_k} f(I)|^2 \\
&\geqslant \sum_{1 \leqslant i_m \leqslant d_n} |\tilde{\eta}_{i_1} \cdots \tilde{\eta}_{i_k} f(I)|^2 = |(D_n^k f)(I)|_{t,n}^2.
\end{aligned}
$$

Therefore $\|(1-D_n)_I^{-1} f\|_{t,n}^2 \leqslant \|(1-D_{n+1})_I^{-1} f\|_{t,n+1}^2$, so the claim holds. ∎

Notation 4.2. Let $\mathfrak{h}$ be a complex Lie algebra with a Hermitian inner product on it. Then $T(\mathfrak{h})$ will denote the algebraic tensor algebra over $\mathfrak{h}$ as a complex vector space and $T'(\mathfrak{h})$ will denote the algebraic dual of $T(\mathfrak{h})$. Define a norm on $T(\mathfrak{h})$ by

$$
|\beta|_t^2 = \sum_{k=0}^{n} \frac{k!}{t^k} |\beta_k|^2, \qquad \beta = \sum_{k=0}^{n} \beta_k, \quad \beta_k \in \mathfrak{h}^{\otimes k}, \qquad k = 0, 1, 2, ..., t > 0. \tag{4.1}
$$

Here $|\beta_k|$ is the cross norm on $\mathfrak{h}^{\otimes k}$ arising from the inner product on $\mathfrak{h}^{\otimes k}$ determined by the norm $|\cdot|$ on $\mathfrak{h}$. $T_t(\mathfrak{h})$ will denote the completion of $T(\mathfrak{h})$ in this norm. The topological dual of $T_t(\mathfrak{h})$ may be identified with the subspace $T_t^*(\mathfrak{h})$ of $T'(\mathfrak{h})$ consisting of such $\alpha T'(\mathfrak{h})$ that the norm

$$
|\alpha|_t^2 = \sum_{k=0}^{\infty} \frac{t^k}{k!} |\alpha_k|^2, \qquad \alpha = \sum_{k=0}^{\infty} \alpha_k, \quad \alpha_k \in (\mathfrak{h}^{\otimes k})^*, \qquad k = 0, 1, 2, ..., t > 0, \tag{4.2}
$$

is finite. Here $|\alpha_k|$ is the norm on $(\mathfrak{h}^{\otimes k})^*$ dual to the cross norm on $\mathfrak{h}^{\otimes k}$.

There is a natural pairing for any $\alpha \in T'(\mathfrak{h})$ and $\beta \in T(\mathfrak{h})$ denoted by

$$
\langle \alpha, \beta \rangle = \sum_{k=0}^{\infty} \langle \alpha_k, \beta_k \rangle, \qquad \alpha = \sum_{k=0}^{\infty} \alpha_k, \qquad \beta = \sum_{k=0}^{n} \beta_k,
$$

$$
\alpha_k \in (\mathfrak{h}^{\otimes k})^*, \quad \beta_k \in \mathfrak{h}^{\otimes k}, \quad k = 0, 1, 2, ...
$$

Denote by $J(\mathfrak{h})$ the two-sided ideal in $T(\mathfrak{h})$ generated by $\{\xi\otimes\eta-\eta\otimes\xi-[\xi,\eta],\ \xi,\ \eta\in\mathfrak{h}\}$. Let $J^0(\mathfrak{h})=\{\alpha\in T'(\mathfrak{h}):\alpha(J)=0\}$. Finally, let $J^0_t(\mathfrak{h})=T^*_t(\mathfrak{h})\cap J^0(\mathfrak{h})$. We will denote $J^0_t(\mathfrak{g})$ by J^0_t, $J^0_t(\mathfrak{g}_n)$ by $J^0_{t,n}$.

The coefficients $k!/t^k$ in the norm on the tensor algebra $T(\mathfrak{h})$ are related to the heat kernel.

By Theorem 3.6 all D^k_n have continuous extensions from $\bigcup_n G_n$ to G_∞, which allows us to introduce the following definition.

Definition 4.3. $\mathscr{H}^t(G_\infty)$ is a space of continuous functions on G_∞ such that their restrictions to G_n are holomorphic for every n and $\|f\|_{t,\infty}=\sup_n\{\|f\|_{t,n}\}=\lim_{n\to\infty}\|f\|_{t,n}<\infty$.

Theorem 4.4. *$\mathscr{H}^t(G_\infty)$ is a Hilbert space and $(1-D)_I^{-1}$ is an isometry from $\mathscr{H}^t(G_\infty)$ into J^0_t.*

Proof. First let us show that $(1-D)_I^{-1}$ is an isometry. T_n is a subalgebra of T. Note that T'_n can be easily identified with a subspace of T'. Namely, for any $\alpha_n\in T'_n$ we can define α as follows

$$\alpha=\begin{cases}\alpha_n & \text{on } T_n\\ 0 & \text{on } T_n^\perp\end{cases}.$$

Therefore $T'_n=(T_n^\perp)^0$. Define Π_n to be an orthogonal projection from T to T_n. Let Π'_n denote the following map from T' to T'_n: $(\Pi'_n\alpha)(x)=\alpha(\Pi_n x)$, $\alpha\in T'$, $x\in T$. Then $\Pi'_n\circ(1-D)_I^{-1}:\mathscr{H}L^2(G_\infty)\to T'_n$ is equal to $(1-D_n)_I^{-1}$. Indeed, note that if we choose an orthonormal basis of the complex inner product space $\mathfrak{g}$ such that $\{\eta_m\}_{m=1}^{d_n}$ is an orthonormal basis of $\mathfrak{g}_n$, then Π_n can be described explicitly,

$$\Pi_n(\eta_{k_1}\otimes\eta_{k_2}\otimes\cdots\otimes\eta_{k_l})=\begin{cases}0 & \text{if}\quad k_s>d_n \qquad \text{for some}\quad 1\leqslant s\leqslant l\\ \eta_{k_1}\otimes\eta_{k_2}\otimes\cdots\otimes_{k_l} & \text{otherwise,}\end{cases}$$

and

$$\begin{aligned}&\langle\Pi'_n\circ(1-D)_I^{-1}f,\eta_{k_1}\otimes\eta_{k_2}\otimes\cdots\otimes\eta_{k_l}\rangle\\ &\quad=\langle(1-D)_I^{-1}f,\Pi_n(\eta_{k_1}\otimes\eta_{k_2}\otimes\cdots\otimes\eta_{k_l})\rangle\\ &\quad=\begin{cases}0 & \text{if}\quad k_s>d_n \qquad \text{for some}\quad 1\leqslant s\leqslant l\\ \langle(1-D)_I^{-1}f,\eta_{k_1}\otimes\eta_{k_2}\otimes\cdots\otimes\eta_{k_l}\rangle & \text{otherwise.}\end{cases}\end{aligned}$$

So $\Pi'_n\circ(1-D)_I^{-1}=(1-D_n)_I^{-1}$. Driver and Gross proved in [5] that $\Pi'_n\circ(1-D)_I^{-1}$ is an isometry from $\mathscr{H}L^2(G_n,\mu^n_t)$ into $J^0_{t,n}$. Let us define a restriction map $R_n\colon \mathscr{H}L^2(G_\infty) \to \mathscr{H}L^2(G_n, \mu_t^n)$ by $f \mapsto f|_{G_n}$. Thus we have a commutative diagram:

$$\begin{array}{ccc} \mathscr{H}L^2(G_\infty) & \xrightarrow{(1-D)_I^{-1}} & J_t^0 \\ {\scriptstyle R_n}\downarrow & & \downarrow{\scriptstyle \Pi'_n} \\ \mathscr{H}L^2(G_n, \mu_t^n) & \xrightarrow[\text{Driver–Goss}]{(1-D_n)_I^{-1}} & J_{t,n}^0 \end{array} \tag{4.3}$$

Now we can prove that $(1-D)_I^{-1}$ is an isometry. By Lemma 4.1.

$$\|f\|_{t,\infty} = \lim_{n\to\infty} \|f\|_{t,n} = \lim_{n\to\infty} \|R_n f\|_{t,n}.$$

It is clear that $|\Pi'_n\alpha|_t = |\Pi'_n\alpha|_{t,n} \xrightarrow[n\to\infty]{} |\alpha|_t$ for any $\alpha \in T'$. In particular, $|\Pi'_n \circ (1-D)_I^{-1} f|_{t,n} \xrightarrow[n\to\infty]{} |(1-D)_I^{-1} f|_t$. At the same time $|\Pi'_n \circ (1-D)_I^{-1} f|_{t,n} = |(1-D_n)_I^{-1} \circ R_n f|_{t,n} = \|R_n f\|_{t,n} \xrightarrow[n\to\infty]{} \|f\|_{t,\infty}$ by the Driver–Gross isomorphism.

Now let us check that $\mathscr{H}^t(G_\infty)$ is a Hilbert space. It is clear that $\|\cdot\|_{t,\infty}$ is a seminorm. Suppose that $\|f\|_{t,\infty} = 0$. Then $\|f\|_{t,n} = 0$ for any n, $t > 0$. We know that $\mathscr{H}L^2(G_n, \mu_t^n)$ is a Hilbert space, therefore $f|_{G_n} = 0$ for all n. By Lemma 3.6 we have that $f|_{G_\infty} = 0$. Thus $\|\cdot\|_{t,\infty}$ is a norm.

Let us now show that $\mathscr{H}^t(G_\infty)$ is a complete space. Suppose $\{f_m\}_{m=1}^\infty$ is a Cauchy sequence in $\mathscr{H}^t(G_\infty)$. Then $\{f_m|_{G_n}\}_{m=1}^\infty$ is a Cauchy sequence in $\mathscr{H}L^2(G_n, \mu_t^n)$ for all n. Therefore there exists g_n from $\mathscr{H}L^2(G_n, \mu_t^n)$ such that $f_m|_{G_n} \xrightarrow[m\to\infty]{} g_n$. Note that $g_{n+m}|_{G_n} = g_n$. In addition,

$$\|g_n\|_{t,n} \leqslant \|f_m|_{G_n}\|_{t,n} + \|f_m|_{G_n} - g_n\|_{t,n} \leqslant \|f_m\|_{t,\infty} + \|f_m|_{G_n} - g_n\|_{t,n}.$$

Note that $\{\|f_m\|_{t,\infty}\}_{m=1}^\infty$ is again a Cauchy sequence, so it has a (finite) limit as $m \to \infty$. Taking a limit in (4) as $m \to \infty$ we get that $\{\|g_n\|_{t,n}\}_{n=1}^\infty$ are uniformly bounded.

By Lemma 3.6 there exists a continuous function g on G_∞ such that $g|_{G_n} = g_n$. Thus $g \in \mathscr{H}^t(G_\infty)$. Now we need to prove that $f_m \xrightarrow[m\to\infty]{} g$ in $\mathscr{H}^t(G_\infty)$. Let $\alpha = (1-D)_I^{-1} g$, $\alpha_m = (1-D)_I^{-1} f_m$. As we have shown $(1-D)_I^{-1}$ is an isometry, so α_m is a Cauchy sequence in J_t^0. Thus there exists $\alpha' \in J_t^0$ such that $\alpha_m \xrightarrow[m\to\infty]{} \alpha'$ (in J_t^0). The question is whether $\alpha = \alpha'$. Note that

$$\Pi'_n\alpha = (1-D_n)_I^{-1} g \qquad \text{and} \qquad \Pi'_n\alpha_m = (1-D_n)_I^{-1} f_m.$$

We know that $\mathscr{H}L^2(G_n, \mu_t^n)$ is a Hilbert space; therefore $\Pi'_n\alpha = \Pi'_n\alpha'$ for any n. Thus $\alpha = \alpha'$, which completes the proof. ∎

THEOREM 4.5. *Suppose connected (finite dimensional) Lie groups G_n are simply connected. Then the map $(1-D)_I^{-1}$ is surjective onto J_t^0.*

Proof. Indeed, for any $\alpha \in J_t^0$, $\Pi'_n \alpha \in J_{t,n}^0$ by the Driver–Gross isomorphism there exists a unique $f_n \in \mathscr{H}L^2(G_n, \mu_t^n)$ such that $(1-D)_{I,n}^{-1} f_n = \Pi'_n \alpha$. Moreover, $R_n f_{n+1} = f_n$ by the commutativity of diagram 4.3 and uniqueness of functions f_n. Indeed, from the diagram we have $(1-D_n)_I^{-1} (R_n f_{n+1}) = \Pi'_n \circ (1-D)_I^{-1} f_{n+1} = (\Pi'_n \circ \Pi'_{n+1}) \circ (1-D)_I^{-1} f_{n+1} = \Pi'_n \circ (\Pi'_{n+1} \circ (1-D)_I^{-1}) f_{n+1} = \Pi'_n \alpha$ by the definition of f_{n+1}. Thus we can define f on $\bigcup_n G_n$ as follows: $f|_{G_n} = f_n$. By Theorem 3.6 there is a unique continuous function g on G_∞ such that $g|_{G_\infty} = f$. Note that $\|f\|_{t,n} = \|f_n\|_{t,n} = |\alpha|_t$; therefore by Proposition 4.1 $\|g\|_{t,\infty} < \infty$ and so $g \in \mathscr{H}L^2(G_\infty)$. ∎

Here we should note that sometimes $(1-D)_I^{-1}$ is an isometry onto a trivial space. Indeed, in [7] we proved the following theorem.

THEOREM 4.6. *Suppose $\mathfrak{g}$ is a Lie algebra with an inner product $\langle \cdot, \cdot \rangle$. Assume that there is an orthonormal basis $\{\xi_k\}_{k=1}^\infty$ of $\mathfrak{g}$ such that for any k there are nonzero $\alpha_k \in \mathbb{C}$ and an infinite set of distinct pairs (i_m, j_m) satisfying $\xi_k = \alpha_k [\xi_{i_m}, \xi_{j_m}]$. Then J_t^0 is isomorphic to $\mathbb{C}$.*

In particular, the conclusion of this theorem holds for a Lie algebra of the Hilbert–Schmidt complex orthogonal group SO_{HS} and a Lie algebra of the Hilbert–Schmidt complex symplectic group $\mathrm{Sp}_{\mathrm{HS}}$ with invariant inner product (see Section 9). One of the ways to deal with this problem is to show that there are non constant functions in $\mathscr{H}^t(G_\infty)$. It will be done in Section 8, but before we can manage this we need to construct the heat kernel measure.

5. THE HEAT KERNEL MEASURE

Suppose $G_n \subset I + HS$, $\mathfrak{g}_n \subset HS$, and $|\cdot| \geqslant |\cdot|_{\mathrm{HS}}$. Then $\mathfrak{g}_\infty$ is embedded in HS. There is a quadratic functional $\tilde{Q}$ on $\mathfrak{g}_\infty$ defined by $\tilde{Q}(x) = \|x\|_{\mathrm{HS}}^2$. Thus there exists a positive operator Q: $\mathfrak{g}_\infty \to \mathfrak{g}_\infty$ such that

$$(x, Qx)_{\mathfrak{g}_\infty} = \tilde{Q}(x) = \|x\|_{\mathrm{HS}}^2 = (x, x)_{\mathrm{HS}}.$$

Note that Q is a bounded complex linear operator. The operator Q will be identified with its nonnegative extension to HS by $Q|_{(\mathfrak{g}_\infty)_{\mathrm{HS}}^\perp} = 0$. We assume that Q is a trace class operator on HS and that all $\mathfrak{g}_n$ are its invariant subspaces.

We begin with the definition of the process Y_t. Let W_t be a $\mathfrak{g}_\infty$-valued Wiener process with a covariance operator Q: $HS \to \mathfrak{g}_\infty$.

In what follows let $\{\xi_n\}_{n=1}^{\infty}$ denote an orthonormal basis of $\mathfrak{g}_\infty$ as a real space such that $\{\xi_n\}_{n=1}^{2d_n}$ is an orthonormal basis of $\mathfrak{g}_n$. Here $d_n = \dim_{\mathbb{C}} \mathfrak{g}_n$, the complex dimension of $\mathfrak{g}_n$. Then $\{e_n\} = \{Q^{-1/2}\xi n\}_{n=1}^{\infty}$ is an orthonormal basis of the $\mathrm{Span}_{\mathrm{HS}}(\mathfrak{g}_\infty)$.

Denote by $L_2^0 = L_2(\mathfrak{g}_\infty, HS)$ the space of the Hilbert–Schmidt operators from $\mathfrak{g}_\infty$ to HS with the (Hilbert–Schmidt) norm $\|\Psi\|_{L_2^0}^2$. Let B: $HS \to L_2^0$, $B(Y)\, U = (Y+I)\, U$ for $U \in \mathfrak{g}_\infty$; then the following theorem holds.

THEOREM 5.1. 1. *The stochastic differential equation*

$$\begin{aligned} dY_t &= B(Y_t)\, dW_t, \\ Y_0 &= 0 \end{aligned} \tag{5.1}$$

has a unique solution, up to equivalence, among the processes satisfying

$$\mathbf{P}\left(\int_0^T \|Y_s\|_{\mathrm{HS}}^2\, ds < \infty\right) = 1.$$

2. *For any $p \geqslant 2$ there exists a constant $C_{p,T} > 0$ such that*

$$\sup_{t \in [0, T]} \mathbf{E} \|Y_t\|_{\mathrm{HS}}^p \leqslant C_{p,T}.$$

Proof of Theorem 5.1. To prove this theorem we will use Theorem 7.4, from the book by DaPrato and Zabczyk [3, p. 186]. It is enough to check that

1. $B(Y)$ is a measurable mapping from HS to L_2^0.
2. $\|B(Y_1) - B(Y_2)\|_{L_2^0} \leqslant C \|Y_1 - Y_2\|_{\mathrm{HS}}$ for any $Y_1, Y_2 \in HS$.
3. $\|B(Y)\|_{L_2^0}^2 \leqslant K(1 + \|Y\|_{\mathrm{HS}}^2)$ for any $Y \in HS$.

Proof of 1. We want to check that $B(Y)$ is in L_2^0 for any Y from HS. First of all, $B(Y)\, U \in HS$, for any $U \in \mathfrak{g}_\infty$. Indeed, $B(Y)\, U = (Y+I)\, U = YU + U \in HS$, since U and Y are in HS.

Now let us verify that $B(Y) \in L_2^0$. Consider the Hilbert–Schmidt norm of B as an operator from $\mathfrak{g}_\infty$ to HS. Then the Hilbert–Schmidt norm of B can be found as follows

$$\begin{aligned} \|B(Y)\|_{L_2^0}^2 &= \sum_{n=1}^{\infty} \langle B(Y)\,\xi_n, B(Y)\,\xi_n\rangle_{\mathrm{HS}} = \sum_{n=1}^{\infty} \langle (Y+I)\,\xi_n, (Y+I)\,\xi_n\rangle_{\mathrm{HS}} \\ &\leqslant \|Y+I\|^2 \sum_{n=1}^{\infty} \langle \xi_n, \xi_n\rangle_{\mathrm{HS}} = \|Y+I\|^2 \sum_{n=1}^{\infty} \langle e_1, Qe_n\rangle_{\mathrm{HS}} \\ &= \|Y+I\|^2 \operatorname{Tr} Q < \infty, \end{aligned}$$

since the operator norm $\|Y+I\|$ is finite.

Proof of 2. Similarly to the previous proof we have

$$\|B(Y_1)-B(Y_2)\|_{L_2^0}\leqslant \|Y_1-Y_2\|_{\mathrm{HS}}\,(\operatorname{Tr} Q)^{1/2}.$$

Proof of 3. Use the estimate in the proof of 1.

$$\|B(Y)\|_{L_2^0}\leqslant (\operatorname{Tr} Q)^{1/2}\,\|Y+I\|\leqslant (\operatorname{Tr} Q)^{1/2}\,(\|Y\|_{\mathrm{HS}}+1), \qquad \text{so}$$
$$\|B(Y)\|^2_{L_2^0}\leqslant 2(\operatorname{Tr} Q)(1+\|Y\|^2_{\mathrm{HS}}). \quad \blacksquare$$

LEMMA 5.2. $\sum_{n=1}^{\infty}\xi_n^2=0$.

Proof. First let us check that $\sum_{n=1}^{\infty}\xi_n^2$ does not depend on the choice of the orthonormal basis $\{\xi_n\}_1^{\infty}$ in $\mathfrak{g}_\infty$. Define a bilinear real form on $HS\times HS$ by $L(f,g)=\Lambda(Q^{1/2}fQ^{1/2}g)$, where Λ is a real bounded linear functional on HS. Then $f\mapsto L(f,g)$ is a bounded linear functional on HS and so $L(f,g)=\langle f,\tilde g\rangle_{\mathrm{HS}}$ for some $\tilde g\in HS$. There exists a linear operator B on HS such that $L(f,g)=\langle f,Bg\rangle_{\mathrm{HS}}$. We can actually compute B. Indeed, there exists an element $h\in HS$ such that $\Lambda(x)=\langle x,h\rangle_{\mathrm{HS}}$. Then

$$L(f,g)=\langle Q^{1/2}fQ^{1/2}g,h\rangle_{\mathrm{HS}}$$

and so

$$\begin{aligned}L(f,g)&=\operatorname{Tr}(Q^{1/2}fQ^{1/2}gh^*)=\langle Q^{1/2}f,h(Q^{1/2}g)^*\rangle_{\mathrm{HS}}\\&=\langle f,Q^{1/2}(h(Q^{1/2}g)^*)\rangle_{\mathrm{HS}}.\end{aligned}$$

Thus $Bg=Q^{1/2}(h(Q^{1/2}g)^*)$ for some $h\in HS$. Therefore B is trace class and since trace is independent of a basis

$$\operatorname{Tr} B=\sum_{n=1}^{\infty}\langle e_n,Be_n\rangle_{\mathrm{HS}}=\sum_{n:\,e_n\in\mathfrak{g}_\infty}\Lambda(Q^{1/2}e_nQ^{1/2}e_n)=\Lambda\left(\sum_{n=1}^{\infty}\xi_n^2\right)$$

does not depend on the choice of $\{\xi_n\}_{n=1}^{\infty}$ for any Λ. Now choose $\{\xi_n\}_{n=1}^{\infty}$ so that $\xi_{2n}=i\xi_{2n-1}$ for $n=1,2,\ldots$. Here $i=\sqrt{-1}$. Then $(\xi_{2n-1})^2+(\xi_{2n})^2=0$. $\blacksquare$

Remark 5.3. In case when $\mathfrak{g}_\infty$ is a real space without complex structure, the process is a solution of the equation

$$dY_t=B(Y_t)\,dW_t+\sum_{n=1}^{\infty}\xi_n^2(Y_t+I)\,dt,$$
$$Y_0=0.$$

The dt term is necessary to ensure that the generator of Y_t is the Laplacian.

THEOREM 5.4. *$Y_t + I \in GL(H)$ for any $t > 0$ with probability* 1. *The inverse of $Y_t + I$ is $\tilde{Y}_t + I$, where $\tilde{Y}_t$ is a solution to the stochastic differential equation*

$$\begin{aligned} d\tilde{Y}_t &= \tilde{B}(\tilde{Y}_t)\, dW_t, \\ \tilde{Y}_0 &= 0, \end{aligned} \tag{5.2}$$

where $\tilde{B}(Y)(U) = -U(Y+I)$, $U \in HS$.

Proof of Theorem 5.4. First we will check that $(\tilde{Y}_t + I)(Y_t + I) = I$ with probability 1 for any $t > 0$. To do this we will apply Itô's formula to $G(Y_t, \tilde{Y}_t)$, where G is defined as follows: $G(\mathbf{Y}) = \Lambda((Y_1 + I)(Y_2 + I))$, where $\mathbf{Y} = (Y_1, Y_2) \in HS \times HS$, and Λ is a nonzero linear real bounded functional from $HS \times HS$ to $\mathbb{R}$. Here we view G as a function on a Hilbert space $HS \times HS$ and consider $G(\mathbf{Y}_t) = G(Y_t, \tilde{Y}_t)$. Then $(Y_t + I)(\tilde{Y}_t + I) = I$ if and only if $\Lambda((Y_t + I)(\tilde{Y}_t + I) - I) = 0$ for any Λ. In order to use Itô's formula we must verify several properties of the processes $\tilde{Y}_t$ and the mapping G:

1. $\tilde{B}(\tilde{Y}_s)$ is an L_2^0-valued process stochastically integrable on $[0, T]$.
2. G and the derivatives G_t, $G_{\mathbf{Y}}$, $G_{\mathbf{YY}}$ are uniformly continuous on bounded subsets of $[0, T] \times HS \times HS$.

Proof of 1. See 1 in the proof of Theorem 5.1.

Proof of 2. Let us calculate G_t, $G_{\mathbf{Y}}$, $G_{\mathbf{YY}}$. First, $G_t = 0$. For any $\mathbf{S} \in HS \times HS$,

$$G_{\mathbf{Y}}(\mathbf{Y})(\mathbf{S}) = \Lambda(S_1(Y_2 + I) + (Y_1 + I)\, S_2).$$

For any $\mathbf{S}, \mathbf{T} \in HS \times HS$

$$G_{\mathbf{YY}}(\mathbf{Y})(\mathbf{S} \otimes \mathbf{T}) = \Lambda(S_1 T_2 + T_1 S_2).$$

Thus condition 2 is satisfied.

We will use the notation

$$G_{\mathbf{Y}}(\mathbf{Y})(\mathbf{S}) = \langle \bar{G}_{\mathbf{Y}}(\mathbf{Y}), \mathbf{S} \rangle_{\mathbf{HS}},$$

$$G_{\mathbf{YY}}(\mathbf{Y})(\mathbf{S} \otimes \mathbf{T}) = \langle \bar{G}_{\mathbf{YY}}(\mathbf{Y})\, \mathbf{S}, \mathbf{T} \rangle_{\mathbf{HS}},$$

where $\bar{G}_{\mathbf{Y}}$ is an element of $HS \times HS$ and $\bar{G}_{\mathbf{YY}}$ is an operator on $HS \times HS$ corresponding to the functionals $G_{\mathbf{Y}} \in (HS \times HS)^*$ and $G_{\mathbf{YY}} \in ((HS \times HS) \otimes (HS \times HS))^*$.

Now we can apply Itô's formula to $G(\mathbf{Y}_t)$:

$$G(\mathbf{Y}_t) = \int_0^t \langle \bar{G}_{\mathbf{Y}}(\mathbf{Y}_s), (B(Y_s)\, dW_s, \tilde{B}(\tilde{Y}_s)\, dW_s) \rangle_{\mathbf{HS}}$$

$$+ \int_0^t \tfrac{1}{2} \mathrm{Tr}_{\mathbf{HS}}[\bar{G}_{\mathbf{YY}}(\mathbf{Y}_s)(B(Y_s)\, Q^{1/2}, (\tilde{B}(\tilde{Y}_s)\, Q^{1/2})^*)]\, ds. \quad (5.3)$$

Let us calculate the two integrands in (5.3) separately. The first integrand is

$$\begin{aligned}
&\langle \bar{G}_{\mathbf{Y}}(\mathbf{Y}_s), (B(Y_s)\, dW_s, \tilde{B}(\tilde{Y}_s)\, dW_s) \rangle_{\mathbf{HS}} \\
&\quad = G_{\mathbf{Y}}(\mathbf{Y}_s)((Y_s + I)\, dW_s, -dW_s(\tilde{Y}_s + I)) \\
&\quad = \Lambda((Y_s + I)\, dW_s(\tilde{Y}_s + I) - (Y_s + I)\, dW_s(\tilde{Y}_s + I)) = 0.
\end{aligned}$$

The second integrand is

$$\begin{aligned}
&\tfrac{1}{2} \mathrm{Tr}_{\mathbf{HS}}[\bar{G}_{\mathbf{YY}}(\mathbf{Y}_s)((B(Y_s)\, Q^{1/2}, \tilde{B}(\tilde{Y}_s)\, Q^{1/2})(B(Y_s)\, Q^{1/2}, \tilde{B}(\tilde{Y}_s)\, Q^{1/2})^*)] \\
&\quad = \tfrac{1}{2} \sum_{n=1}^{\infty} G_{\mathbf{YY}}(\mathbf{Y}_s)((B(Y_s)\, Q^{1/2} e_n, \tilde{B}(\tilde{Y}_s)\, Q^{1/2} e_n) \\
&\qquad \otimes (B(Y_s)\, Q^{1/2} e_n, \tilde{B}(\tilde{Y}_s)\, Q^{1/2} e_n)) \\
&\quad = \tfrac{1}{2} \sum_{n=1}^{\infty} \Lambda(-(Y_s + I)\, \xi_n^2(\tilde{Y}_s + I) - (Y_s + I)\, \xi_n^2(\tilde{Y}_s + I)) \\
&\quad = \sum_{n=1}^{\infty} \Lambda(-(Y_s + I)\, \xi_n^2(\tilde{Y}_s + I)) = 0
\end{aligned}$$

by Lemma 5.2. This shows that the stochastic differential of G is zero, so $G(\mathbf{Y}_t) = I$ for any $t > 0$. By the Fredholm alternative $Y_t + I$ has an inverse; therefore it has to be $(\tilde{Y}_t + I)$. ∎

In some cases we can prove that the process $Y_t + I$ lives in a smaller group. For example, it can be shown if the group is defined by a relatively simple relation (see Section 9).

Let us define μ_t as

$$\int_{I + HS} f(X)\, \mu_t(dX) = Ef(X_t(I)) = P_{t,0} f(I)$$

for any bounded Borel function f on $I + HS$.

DEFINITION 5.5. μ_t is called *the heat kernel measure on* G_∞. The space of all square integrable functions on $I+HS$ is denoted by $L^2(I+HS, \mu_t)$ and the corresponding norm by $\|f\|_{L^2(I+HS,\mu_t)} = \|f\|_t$.

A motivation for such a name for μ_t is the fact that Kolmogorov's backward equation corresponding to the process Y_t+I is the heat equation in a sense. First of all, the coefficient B depends only on $Y \in HS$; therefore $P_{s,t}f(Y) = Ef(Y(t, s; Y)) = P_{t-s}f(Y)$. According to Theorem 9.16 from [3], for any $\varphi \in C_b^2(HS)$ and $Y \in HS$, the function $v(t, Y) = P_t\varphi(Y)$ is a unique strict solution from $C_b^{1,2}(HS)$ for the parabolic type equation (Kolmogorov's backward equation)

$$\begin{aligned} &\frac{\partial}{\partial t} u(t, Y) = \frac{1}{2} \operatorname{Tr}[v_{YY}(t, Y)(B(Y)\,Q^{1/2})(B(Y)\,Q^{1/2})^*] \\ &v(0, Y) = \varphi(Y), \qquad t > 0, \quad Y \in HS. \end{aligned} \tag{5.4}$$

Here $C_b^n(HS)$ denotes the space of all functions from HS to $\mathbb{R}$ that are n-times continuously Fréchet differentiable with all derivatives up to order n bounded and $C_b^{k,n}(HS)$ denotes the space of all functions from $[0, T] \times HS$ to $\mathbb{R}$ that are k-times continuously Fréchet differentiable withe respect to t and n-times continuously Fréchet differentiable with respect to Y with all partial derivatives continuous in $[0, T] \times HS$ and bounded.

Let us rewrite Eq. (5.4) as the heat equation. First

$$\begin{aligned} &\operatorname{Tr}[v_{YY}(t, Y)(B(Y)\,Q^{1/2})(B(Y)\,Q^{1/2})^*] \\ &\quad = \sum_{n=1}^{\infty} v_{YY}(t, Y)(B(Y)\,Q^{1/2}e_n \otimes B(Y)\,Q^{1/2})e_n \\ &\quad = \sum_{n=1}^{\infty} v_{YY}(t, Y)((Y+I)\,Q^{1/2}e_n \otimes (Y+I)\,Q^{1/2}e_n) \\ &\quad = \sum_{n=1}^{\infty} v_{YY}(t, Y)((Y+I)\,\xi_n \otimes (Y+I)\,\xi_n), \end{aligned}$$

where $v_{YY}(t, Y)$ is viewed as a functional on $HS \otimes HS$. Now change Y to $X-I$. Then for any smooth bounded function $\varphi(X)\colon I+HS \to \mathbb{R}$, the function $v(t, X) = P_t\varphi(X)$ satisfies this equation, which can be considered as the heat equation,

$$\begin{aligned} &\frac{\partial}{\partial t} v(t, X) = L_1 v(t, X) \\ &v(0, X) = \varphi(X), \qquad t > 0, \qquad X \in I+HS, \end{aligned} \tag{5.5}$$

where the differential operator L_1 on the space $C_b^{1,2}(I+HS)$ is defined by

$$L_1 v \stackrel{\text{def}}{=} \tfrac{1}{2} \sum_{n=1}^{\infty} v_{YY}(t, Y)((Y+I)\,\xi_n \otimes (Y+I)\,\xi_n).$$

Our goal is to show that L_1 is a Laplacian in a sense. More precisely, L_1 is a half of the sum of second derivatives in the directions of an orthonormal basis of $\mathfrak{g}_\infty$. Define the Laplacian by

$$(\Delta v)(X) = \tfrac{1}{2} \sum_{n=1}^{\infty} (\tilde{\xi}_n \tilde{\xi}_n v)(X), \tag{5.6}$$

where $(\tilde{\xi}_n v)(X) = d/dt|_{t=0}\, v(Xe^{t\xi_n})$ for a function $v\colon I+HS \to \mathbb{R}$ and so $\tilde{\xi}_n$ is the left-invariant vector field on $GL(H)$ corresponding to ξ_n.

Let us calculate derivatives of $v\colon I+HS \to \mathbb{R}$ in the direction of ξ_n,

$$(\tilde{\xi}_n v)(X) = v_X(X) \left.\frac{d}{dt}\right|_{t=0} (Xe^{t\xi_n}) = v_X(X)(X\xi_n)$$

and therefore

$$(\tilde{\xi}_n \tilde{\xi}_n v)(X) = v_{XX}(X)(X\xi_n \otimes X\xi_n) + v_X(X\xi_n^2).$$

Thus the Laplacian is

$$\begin{aligned}(\Delta v)(X) &= \tfrac{1}{2} \sum_{n=1}^{\infty} [v_{XX}(X)(X\xi_n \otimes X\xi_n) + v_X(X\xi_n^2)] \\ &= \tfrac{1}{2} \sum_{n=1}^{\infty} v_{XX}(X)(X\xi_n \otimes X\xi_n)\end{aligned}$$

by Lemma 5.2.

Since $\tilde{\xi}_n$ is a left-invariant vector field, the Laplacian Δ is a left-invariant differential operator such that $L_1 v = \Delta v$ for any $v \in C_b^2(I+HS)$.

Proposition 5.6. *For any $p \geqslant 2$, $t > 0$*

$$E\, \|Y_t\|_{\mathrm{HS}}^p < \frac{1}{C_{p,t}} (e^t C_{p,t} - 1),$$

where $C_{p,t} = C_{p/2} 2^{p-1} (\operatorname{Tr} Q)^{p/2}\, t^{(p/2)-1}$, $C_p = (p(2p-1))^p\, (2p/(2p-1))^{2p^2}$.

Proof. First of all, let us estimate $E\, \|\int_0^t B(Y_s)\, dW_s\|_{\mathrm{HS}}^p$. From part 3 of the proof of Theorem 5.1 we know that $\|B(Y)\|_{L_2^0}^2 \leqslant 2 \operatorname{Tr} Q(\|Y\|_{\mathrm{HS}}^2 + 1)$. In addition we will use Lemma 7.2 from the book by DaPrato and Zabczyk

[3, p. 182]: for any $r \geqslant 1$ and for an arbitrary L_2^0-valued predictable process $\Phi(t)$,

$$E\left(\sup_{s\in[0,t]}\left\|\int_0^s \Phi(u)\,dW(u)\right\|_{\mathrm{HS}}^{2r}\right)$$
$$\leqslant C_r E\left(\int_0^t \|\Phi(s)\|_{L_2^0}^2\,ds\right)^r, \qquad t\in[0,T], \tag{5.7}$$

where $C_r=(r(2r-1))^r\,(2r/(r-1))^{2r^2}$. Thus

$$E\left\|\int_0^t B(Y_s)\,dW_s\right\|_{HS}^p$$
$$\leqslant C_{p/2}E\left(\int_0^t \|B(Y_s)\|_{L_2^0}^2\,ds\right)^{p/2}$$
$$\leqslant C_{p/2}(2\operatorname{Tr} Q)^{p/2}\,E\left(\int_0^t (\|Y\|_{\mathrm{HS}}^2+1)\,ds\right)^{p/2}$$
$$\leqslant C_{p/2}2^{p/2}(\operatorname{Tr} Q)^{p/2}\,t^{(p/2)-1}\,E\int_0^t (\|Y\|_{\mathrm{HS}}^2+1)^{p/2}\,ds. \tag{5.8}$$

Now we can use inequality $(x+1)^q \leqslant 2^{q-1}(x^q+1)$ for any $x\geqslant 0$ for the estimate (5.8)

$$E\left\|\int_0^t B(Y_s)\,dW_s\right\|_{\mathrm{HS}}^p$$
$$\leqslant C_{p/2}2^{p/2}(\operatorname{Tr} Q)^{p/2}\,t^{(p/2)-1}2^{(p/2)-1}E\int_0^t (1+\|Y_s\|_{\mathrm{HS}}^p)\,ds$$
$$= C_{p/2}(\operatorname{Tr} Q)^{p/2}\,2^{p-1}t^{(p/2)-1}\left(t+E\int_0^t \|Y\|_{\mathrm{HS}}^p\,ds\right).$$

Finally,

$$E\|Y_t\|_{\mathrm{HS}}^p \leqslant E\left\|\int_0^t B(Y_s)\,dW_s\right\|_{\mathrm{HS}}^p \leqslant C_{p,t}\left(t+E\int_0^t \|Y_s\|_{\mathrm{HS}}^p\,ds\right),$$

where $C_{p,t}=C_{p/2}2^{p-1}(\operatorname{Tr} Q)^{p/2}\,t^{(p/2)-1}$.

Thus, $E\|Y_t\|_{\mathrm{HS}}^p < 1/C_{p,t}(e^t C_{p,t}-1)$ by Gronwall's lemma. ∎

LEMMA 5.7. *Let $f: I+HS\to[0,\infty]$ be a continuous function in $L^2(I+HS,\mu_t)$. If $\sup_n \|f\|_{t,n}<\infty$ for all n, then $Ef(X_n)\xrightarrow[n\to\infty]{} Ef(X)$.*

Proof. Note that there exists a subsequence $\{X_{n_k}\}$ such that $X_{n_k} \xrightarrow[k\to\infty]{} X$ a.s. We will prove first that if $\sup_k \|f\|_{t, n_k} < \infty$, then $Ef(X_{n_k}) \xrightarrow[k\to\infty]{} Ef(X)$. Denote $g_k(\omega) = f(X_{n_k}(\omega))$, $g(\omega) = f(X(\omega))$, $\omega \in \Omega$.

Our goal is to prove that $\int_\Omega g_k(\omega)\, dP \to \int_\Omega g(\omega)\, dP$ as $k \to \infty$. Define $f_l(X) = \min\{f(X), l\}$ for $l > 0$ and $g_{k,l}(\omega) = f_l(X_{n_k}(\omega))$, $g_l(\omega) = f_l(X(\omega))$. Then $g_{k,l} \leqslant l$, $g_l \leqslant l$ for any $\omega \in \Omega$, so $\int_\Omega g_{k,l}(\omega)\, dP \xrightarrow[k\to\infty]{} \int_\Omega g_l(\omega)\, dP$ by the Dominated Convergence Theorem since f is continuous. By Chebyshev's inequality

$$\begin{aligned}
\int_\Omega (g_k(\omega) - g_{k,l}(\omega))\, dP &= \int_{\{\omega:\, f(X_{n_k}) \geqslant l\}} (f(X_{n_k}) - l)\, dP \\
&\leqslant \int_\Omega f(X_{n_k})\, \mathbb{1}_{\{\omega:\, f(X_{n_k}) \geqslant l\}}\, dP \\
&\leqslant \|f\|_{t, n_k} (P\{\omega:\, f(X_{n_k}) \geqslant l\}) \\
&\leqslant \|f\|_{t, n_k} \frac{E\,|f(X_{n_k})|}{l}.
\end{aligned}$$

Thus

$$0 \leqslant \int_\Omega (g_k(\omega) - g_{k,l}(\omega))\, dP \leqslant \frac{\sup_n \|f\|_{t,n}^2}{l}.$$

Similarly

$$0 \leqslant \int_\Omega (g(\omega) - g_l(\omega))\, dP \leqslant \frac{1}{l} \|f\|_t\, E\,|f(X)|.$$

Therefore $Ef(X_{n_k}) \to Ef(X)$ as $k \to \infty$.

To complete the proof suppose that the conclusion is not true. Then there is a subsequence X_{n_k} such that $|Ef(X_{n_k} - Ef(X)| > \varepsilon$ for any k. However, we always can choose a subsequence $X_{n_{k_m}}$ such that $X_{n_{k_m}} \xrightarrow[m\to\infty]{} X$ a.s. and therefore $Ef(X_{n_{k_m}}) \xrightarrow[m\to\infty]{} Ef(X)$. This is a contradiction. ∎

6. APPROXIMATION OF THE PROCESS

Let P_n be the projection onto $\mathfrak{g}_n$. Note that since we assume that $\mathfrak{g}_n$ are invariant subspaces of Q, the projection from $\mathfrak{g}_\infty$ onto $\mathfrak{g}_n$ (defined in terms of the norm $|\cdot|$) is the restriction of the projection from HS onto $\mathfrak{g}_n$ (defined in terms of the norm $|\cdot|_{\mathrm{HS}}$). Note also that since $\mathfrak{g}_n \cap \operatorname{Ker} Q = \{0\}$, $P_n Q P_n$ is positive and invertible on $\mathfrak{g}_n$ and $(P_n Q P_n)^{-1} = P_n Q^{-1} P_n$.

Choose a real orthonormal basis $\{\xi_m\}_{m=1}^{\infty}$ of $\mathfrak{g}_\infty$ in the same way as it was done in Section 5. Consider an equation

$$dY_{n,t} = B_n(Y_{n,t})\, dW_t, \qquad Y_{n,t}(0) = 0,$$

where

$$B_n(Y)\, U = (Y+I)(P_n U).$$

This equation has a unique solution by the same arguments as in Section 5. Denote $Q_n = P_n Q P_n$.

LEMMA 6.1. *$Y_{n,t}$ is a solution of the equation*

$$\begin{aligned} dY_{n,t} &= B_n(Y_{n,t})\, dW_{n,t}, \\ Y_{n,t}(0) &= 0, \end{aligned} \tag{6.1}$$

where $W_{n,t} = P_n W_t$.

Proof. Check that $P_n Q P_n$ is the covariance operator of $W_{n,t}$. By the definition of a covariance operator we know that $\langle f, Qg\rangle_{HS} = E\langle f, W_1\rangle_{HS} \langle g, W_1\rangle_{HS}$; therefore

$$\begin{aligned} \langle f, Q_n g\rangle_{HS} &= \langle P_n f, QP_n g\rangle_{HS} = E\langle P_n f, W_1\rangle_{HS} \langle P_n g, W_1\rangle_{HS} \\ &= E\,\langle f, P_n W_1\rangle_{HS} \langle g, P_n W_1\rangle_{HS}. \end{aligned}$$

Thus Eq. (6.1) is actually the same as Eq. (5.1) with the Wiener process that has the covariance Q_n instead of Q. ∎

We will use the following lemma from [7].

LEMMA 6.2. 1. $\operatorname{Tr} Q_n \leqslant \operatorname{Tr} Q$.

2. $P_n Q \xrightarrow[n\to\infty]{} Q$,
3. $QP_n \xrightarrow[n\to\infty]{} Q$,
4. $P_n Q P_n \xrightarrow[n\to\infty]{} Q$,

where convergence is in the trace class norm.

THEOREM 6.3. *Denote by $\mathscr{H}_2$ the space of equivalence classes of HS-valued predictable processes with the norm*:

$$|||Y|||_2 = \Big(\sup_{t\in[0,T]} E\,\|Y(t)\|_{HS}^2\Big)^{1/2}.$$

Then

$$|||Y_n - Y|||_2 \xrightarrow[n\to\infty]{} 0.$$

Proof of Theorem 6.3. We want to apply the local inversion theorem (see, for example, Lemma 9.2, from the book by DaPrato and Zabczyk [3, p. 244]) to $K(y, Y) = y + \int_0^t B(Y)\, dW_t$, where y is the initial value of Y and $Y = Y(y, t)$ is an HS-valued predictable process. Analogously we define $K_n(y, Y) = y + \int_0^t B_n(Y)\, dW_t$. To apply this lemma we need to check that K and K_n satisfy the following conditions:

1. For any $Y_1(t)$ and $Y_2(t)$ from $\mathscr{H}_2$

$$\sup_{t\in[0,T]} E\,\|K(y, Y_1) - K(y, Y_2)\|_{\mathrm{HS}}^2 \leqslant \alpha \sup_{t\in[0,T]} E\,\|Y_1(t) - Y_2(t)\|_{\mathrm{HS}}^2,$$

where $0 \leqslant \alpha < 1$.

2. For any $Y_1(t)$ and $Y_2(t)$ from $\mathscr{H}_2$

$$\sup_{t\in[0,T]} E\,\|K_n(y, Y_1) - K_n(y, Y2)\|_{\mathrm{HS}}^2 \leqslant \alpha \sup_{t\in[0,T]} E\,\|Y_1(t) - Y_2(t)\|_{\mathrm{HS}}^2,$$

where $0 \leqslant \alpha < 1$

3. $\lim_{n\to\infty} K_n(y, Y) = K(y, Y)$ in $\mathscr{H}_2$.

Proof of 1. To estimate the part with the stochastic differential we will use Lemma 7.2, [3, p. 182]: for any $r \geqslant 1$ and for arbitrary L_2^0-valued predictable process $\Phi(t)$,

$$E\left(\sup_{s\in[0,t]} \left\|\int_0^s \Phi(u)\, dW(u)\right\|_{\mathrm{HS}}^{2r}\right)$$
$$\leqslant C_r E\left(\int_0^t \|\Phi(s)\|_{L_2^0}^2\, ds\right)^r, \qquad t\in[0,T], \tag{6.2}$$

where $C_r = (r(2r-1))^r\, (2r/(2r-1))^{2r^2}$. Thus

$$\begin{aligned} E\,\|K(y, Y_1) - K(y, Y_2)\|_{\mathrm{HS}}^2 &= E\left\|\int_0^t B(Y_1) - B(Y_2)\, dW_s\right\|_{\mathrm{HS}}^2 \\ &\leqslant 4E \int_0^t \|B(Y_1) - B(Y_2)\|_{L_2^0}^2\, ds \\ &\leqslant 4 \operatorname{Tr} QE \int_0^t \|Y_1 - Y_2\|_{\mathrm{HS}}^2\, ds \\ &\leqslant 4t \operatorname{Tr} Q \sup_{t\in[0,T]} E\,\|Y_1 - Y_2\|_{\mathrm{HS}}^2. \end{aligned}$$

Note that for small t we can make $(2(\mathrm{Tr}\, Q)^2 t + 8\mathrm{Tr}\, Q)\, t$ as small as we wish, therefore 1 holds.

Proof of 2. Similarly to the proof of condition 2 in the proof of Theorem 5.1 and by Lemma 6.2 we have that

$$\begin{aligned}\|B_n(Y_1) - B_n(Y_2)\|^2_{L_0^2} &= \|P_n(\cdot)(Y_1 - Y_2)\|^2_{\mathfrak{g}_\infty \to \mathrm{HS}}\\ &\leqslant \mathrm{Tr}(Q^{1/2} P_n Q^{1/2})\, \|Y_1 - Y_2\|^2_{\mathrm{HS}}\\ &= \mathrm{Tr}(QP_n)\, \|Y_1 - Y_2\|^2_{\mathrm{HS}} \leqslant \mathrm{Tr}\, Q\, \|Y_1 - Y_2\|^2_{\mathrm{HS}}.\end{aligned}$$

Now use the same estimates as in 1 to see that 2 holds.

Proof of 3. Here again we will use (6.2) to estimate the part with the stochastic differential

$$\begin{aligned}|||K_n(y, Y) - K(y, Y)|||^2_2 &= \left|\left|\left|\int_0^t (B_n(Y) - B(Y))\, dW_t\right|\right|\right|^2_2\\ &= \sup_{t \in [0, T]} E \left\|\int_0^t (B_n(Y) - B(Y))\, dW_s\right\|^2_{\mathrm{HS}}\\ &\leqslant 4E \int_0^t \|B_n(Y) - B(Y)\|^2_{L_2^0}\, ds \xrightarrow[n \to \infty]{} 0.\end{aligned}$$

Indeed, estimate $\|B(Y) - B_n(Y)\|^2_{L_2^0}$, as

$$\begin{aligned}\|B(Y) - B_n(Y)\|^2_{L_2^0} &= \sum_{m=1}^{\infty} \|(B(Y) - B_n(Y))\, \xi_m\|^2_{\mathrm{HS}}\\ &= \sum_{m=1}^{\infty} \|(I - P_n)\, \xi_m(Y + I)\|^2_{\mathrm{HS}}\\ &\leqslant \|Y + I\|^2 \sum_{m=1}^{\infty} \|(I - P_n)\, \xi_m\|^2_{\mathrm{HS}}\\ &\leqslant \|Y + I\|^2\, \mathrm{Tr}[Q^{1/2}(I - P_n)\, Q^{1/2}]\\ &= \|Y + I\|^2\, \mathrm{Tr}[(I - P_n)\, Q^{1/2} Q^{1/2}]\\ &= \|Y + I\|^2\, \mathrm{Tr}[(I - P_n)\, Q] \xrightarrow[n \to \infty]{} 0\end{aligned}$$

by Lemma 6.2.

We know that there are unique elements Y, Y_n in the space $\mathscr{H}_2$ such that $Y = K(y, Y)$, $Y_n = K_n(y, Y_n)$, and therefore $\lim_{n \to \infty} Y_n = Y$ for any y by the local inversion lemma. ∎

Denote $P_t^n f(Y) = Ef(Y_n(t, Y))$, $Y \in \mathfrak{g}_n$, and $v^n(t, Y) = P_t^n f(Y)$. Then similarly to 5.4 the following holds for $v^n(t, Y)$.

For any $f \in C_b^2(\mathfrak{g}_n)$ the function $v^n(t, Y)$ is a unique strict solution from $C_b^{1,2}(\mathfrak{g}_n)$ for the parabolic type equation

$$\frac{\partial}{\partial t} v^n(t, Y) = \frac{1}{2} \operatorname{Tr}[v^n_{YY}(B_n(Y)\, Q^{1/2})(B_n(Y)\, Q^{1/2})^*]$$

$$v^n(0, Y) = f(Y), \qquad t > 0, \qquad Y \in \mathfrak{g}_n.$$

We want to show that P_t^n corresponds to the heat kernel measure defined on G_n as on a Lie group (i.e., as in the finite dimensional case).

Note that for any $Y \in \mathfrak{g}_n$.

$$\operatorname{Tr}[v^n_{YY}(t, Y)(B_n(Y)\, Q^{1/2})(B_n(Y)\, Q^{1/2})^*]$$
$$= \sum_{m=1}^{2d_n} v^n_{YY}(t, Y)((Y+I)\, \xi_m \otimes (Y+I)\, \xi_m).$$

Thus

$$L_1 v = \tfrac{1}{2} \operatorname{Tr}[v^n_{YY}(t, Y)(B_n(Y)\, Q^{1/2})(B_n(Y)\, Q^{1/2})^*]$$

is equal to the Laplacian Δ_n on G_n defined similarly to (5.6). Thus, the transition probability P_t^n is equal to $\mu_t^n(dX)$, where the latter is the heat kernel measure on G_n defined in the usual way.

7. CAMERON–MARTIN SUBGROUP

The definition of the Cameron–Martin subgroup G_{CM} was given in Section 2. Let C^1_{CM} denote the space of piecewise differentiable paths h: $[0, 1] \to GL(H)$ such that $h' = h^{-1}\dot{h}$ is in $\mathfrak{g}_\infty$. The main purpose of this section is to prove that $G_{CM} = G_\infty$ under a condition on the Lie bracket. In doing so we also show that under this condition the exponential map is a local diffeomorphism. This implies, in particular, that the group $GL(H)$ is closed in the Riemannian metric induced by the operator norm.

Proposition 7.1. *G_{CM} is a group.*

Proof. Let $x, y \in G_{\mathrm{CM}}$. Take $f[0, 1] \to G_{\mathrm{CM}}$, $f(0) = x$, $f(1) = y$. Define $h(s) = y^{-1} f(s)$. Then $|h^{-1}\dot{h}| = |f^{-1} y y^{-1} \dot{f}| = |f^{-1}\dot{f}|$. Therefore $d(y^{-1}x, I) \leqslant d(x, y) \leqslant d(x, I) + d(y, I) < \infty$, thus $xy^{-1} \in G_{CM}$. ∎

THEOREM 7.2. *If* $|[x, y]| \leqslant C|x|\,|y|$ *for all* $x, y \in \mathfrak{g}_\infty$, *then*

1. $G_{\mathrm{CM}} = G_\infty$.

2. *The exponential map is a diffeomorphism from a neighborhood of* 0 *in* $\mathfrak{g}_\infty$ *onto a neighborhood of* I *in* G_{CM}.

We will need several lemmas before we can prove Theorem 7.2.

LEMMA 7.3. *Take* $g(t)\colon [0, 1] \to G_{\mathrm{CM}}$, $g \in C^1_{\mathrm{CM}}$. *Suppose that* $\|g(t) - I\| < 1$ *for any* $t \in [0, 1]$.

Define $h(t) = \log g(t) = \sum_{n=1}^\infty ((-1)^{n-1}/n)(g(t) - I)^n$. *Denote* $A(t) = g(t)^{-1}\, \dot g(t)$; *then* h *is a unique solution of the ordinary differential equation in* $\mathfrak{g}_\infty$

$$\dot h(t) = F(h, t), \qquad h(0) = 0, \tag{7.1}$$

where

$$F(x, t) = A(t) + \frac{1}{2}[x, A(t)] - \sum_{p=1}^\infty \frac{1}{2(p+2)\,p!}[\cdots[[x, A(t)], \overbrace{x], ..., x}^{p}]. \tag{7.2}$$

Proof. Indeed, $h(t+s) - h(t) = \log g(t+s) - h(t) = \log(g(t)\, g(t)^{-1} g(t+s)) - h(t)$. Denote $f(t, s) = \log(g(t)^{-1}\, g(t+s))$. Then $h(t+s) - h(t) = \log(e^{h(t)} e^{f(t, s)}) - h(t) = DCH(h(t), f(t, s)) - h(t)$, where $DCH(x, y)$ is given by the Dynkin–Campbell–Hausdorff formula for $x, y \in \mathfrak{g}_\infty$.

$$\begin{aligned} DCH(x, y) &= \log(\exp x \exp y) \\ &= \sum_m \sum_{p_t, q_t} \frac{(-1)^{m-1}}{m(\sum_i p_i + q_i)} \\ &\quad \times \frac{[\cdots[\overbrace{[x, x], ..., x]}^{p_1}, \overbrace{y], ..., y]}^{q_1}, ..., \overbrace{y], ..., y]}^{q_m}}{p_1!\, q_1! \cdots p_m!\, q_m!}. \end{aligned} \tag{7.3}$$

Thus

$$\begin{aligned} DCH(h(t), f(t, s)) &= h(t) + f(t, s) + \tfrac{1}{2}[h(t), f(t, s)] \\ &\quad + \tfrac{1}{12}[[h(t), f(t, s)], f(t, s)] \\ &\quad - \tfrac{1}{12}[[h(t), f(t, s)], h(t)] + \cdots. \end{aligned}$$

Let us find $df(t,s)/ds|_{s=0}$

$$\frac{f(t,s+\varepsilon)-f(t,s)}{\varepsilon}$$

$$=\frac{\log(g(t)^{-1}\,g(t+s+\varepsilon))-\log(g(t)^{-1}\,g(t+s))}{\varepsilon}$$

$$=\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}\frac{(g(t)^{-1}\,g(t+s+\varepsilon)-I)^n-(g(t)^{-1}\,g(t+s)-I)^n}{\varepsilon}$$

$$=\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}\frac{1}{\varepsilon}\sum_{k=1}^{n}(g(t)^{-1}\,g(t+s+\varepsilon)-I)^{k-1}$$

$$\times g(t)^{-1}\,(g(t+s+\varepsilon)-g(t+s))(g(t)^{-1}\,g(t+s)-I)^{n-k}$$

$$\xrightarrow[\varepsilon\to\infty]{}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}\sum_{k=1}^{n}(g(t)^{-1}\,g(t+s)-I)^{k-1}\,g(t)^{-1}$$

$$\times\dot{g}(t+s)(g(t)^{-1}\,g(t+s)-I)^{n-k}.$$

Thus

$$\left.\frac{df(t,s)}{ds}\right|_{s=0}$$

$$=\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}\sum_{k=1}^{n}(g(t)^{-1}\,g(t)-I)^{k-1}\,g(t)^{-1}\dot{g}(t)(g(t)^{-1}\,g(t)-I)^{n-k}$$

$$=g(t)^{-1}\,\dot{g}=A(t)$$

$$\frac{h(t+s)-h(t)}{s}$$

$$=\frac{DCH(h(t),f(t,s))-h(t)}{s}$$

$$=\frac{f(t,s)}{s}+\frac{1}{2}\left[h(t),\frac{f(t,s)}{s}\right]+\frac{1}{12}\left[\left[h(t),\frac{f(t,s)}{s}\right],f(t,s)\right]$$

$$+\frac{1}{12}\left[[h(t),f(t,s)],\frac{f(t,s)}{s}\right]-\frac{1}{12}\left[\left[h(t),\frac{f(t,s)}{s}\right],h(t)\right]+\cdots.$$

Note that $f(t, 0)=0$, therefore

$$\begin{aligned}\dot{h}(t) &= A(t)+\frac{1}{2}[h(t), A(t)]\\ &\quad+\frac{d}{ds}\sum_{p}\frac{(-1)^1}{2(p/2)}\frac{[\cdots[[h(t), f(s,t)], \overbrace{h(t)], ..., h(t)]}^{p}}{1!\,1!\,p!\,0!}\bigg|_{s=0}\\ &= A(t)+\frac{1}{2}[h(t), A(t)]\\ &\quad-\sum_{p=1}^{\infty}\frac{1}{2(p+2)\,p!}[\cdots[[h(t), A(t)], \overbrace{h(t)], ..., h(t)]}^{p}.\end{aligned}$$

Note that 7.1 has a unique solution. Indeed, we show later that F satisfies the local Lipshitz conditions

$$|F(x,t)-F(y,t)|\leqslant C\,|x-y|$$

for $|x|<R$, $|y|<R$ and according to Theorem 3.1 of [13], Eq. 7.1 has a unique solution. Indeed, the local Lipshitz conditions are also satisfied for the operator norm. To prove the existence and uniqueness of the solutions in $\mathfrak{g}_\infty$, we first apply Theorem 3.1 of [13] in $B(H)$, then in $\mathfrak{g}_\infty$.

Now let us show that F satisfies the local Lipshitz conditions. Denote

$$G(p,x)=[\cdots[[x, A(t)], \overbrace{x], ..., x]}^{p}$$

Then

$$|F(x,t)-F(y,t)\leqslant\frac{1}{2}|x-y|\,|A_t|+\sum_{p=1}^{\infty}\frac{1}{2(p+2)\,p!}|G(p,x)-G(p,y)|.$$

Let us estimate $|G(p,x)-G(p,y)|$ using induction on p.

When $p=1$,

$$\begin{aligned}|G(1,x)-G(1,y)| &= |[[x, A_t], x]-[[y, A_t], y]|\\ &= |[[x, A_t], x-y]+[[x, A_t]-[y, A_t], y]|\\ &\leqslant |x-y|\,|A_t|\,C(C\,|x|+C\,|y|)\\ &= C^2\,|x-y|\,|A_t|\,(|x|+|y|).\end{aligned}$$

Assume that for $p=k$ we have the estimate

$$|G(k,x)-G(k,y)|\leqslant C^{k+1}\,|x-y|\,|A_t|\sum_{i=0}^{k}|x|^i\,|y|^{k-i}.$$

Then for $p=k+1$ we have

$$\begin{aligned}
&|G(k+1, x)-G(k+1, y)| \\
&\quad = |[G(k, x), x]-[G(k, y), y]| \\
&\quad = |[G(k, x), x]-[G(k, x), y]+[G(k, x), y]-[G(k, y), y]| \\
&\quad \leqslant C\,|G(k, x)|\,|x-y|+C\,|G(k, x)-G(k, y)|\,|y| \\
&\quad \leqslant C^{k+1}\,|x|^{k+2}\,|A_t|\,|x-y|+C^{k+2}\,|x-y|\,|A_t|\left(\sum_{i=0}^{k}|x|^i\,|y|^{k-i}\right)|y| \\
&\quad = C^{k+2}\,|x-y|\,|A_t| \sum_{i=0}^{k+1}|x|^i\,|y|^{k+1-i}.
\end{aligned}$$

Thus if $|x| \leqslant R$, $|y| \leqslant R$

$$|G(p, x)-G(p, y)| \leqslant C^{p+1}\,|x-y|\,|A_t|\,R^p(p+1)$$

and therefore

$$\begin{aligned}
&|F(x, t)-F(y, t)| \\
&\quad \leqslant \frac{1}{2}\,|x-y|\,|A_t|+\sum_{p=1}^{\infty} \frac{1}{2(p+2)\,p!}\,C^{p+1}\,|x-y|\,|A_t|\,R^p(p+1) \\
&\quad \leqslant \frac{1}{2}\,|x-y|\,|A_t|+\frac{1}{2}\,|x-y|\,|A_t|\,C \sum_{p=1}^{\infty} \frac{C^p R^p}{p!} \\
&\quad = \frac{1}{2}\,|x-y|\,|A_t|\,(1+C(e^{\mathrm{CR}}-1)). \quad \blacksquare
\end{aligned}$$

LEMMA 7.4. *There are constants* C_1, C_2, *and* $0<\varepsilon<\ln 2/2C$ *such that*

$$C_1\,|x-y| \leqslant d(\exp x, \exp y) \leqslant C_2\,|x-y|$$

for any $x, y \in \mathfrak{g}_\infty$ *provided* $|x|<\varepsilon$, $|y|<\varepsilon$.

Proof. Take a path $g\colon [0, 1] \rightarrow \exp(\mathfrak{g}_\infty)$ such that $g(0)=e^x$, $g(1)=e^y$. Let $h=\log g$. Then

$$g^{-1}\dot{g}=\sum_{k=0}^{\infty} \frac{(-adh)^k\,(\dot{h})}{(k+1)!}=\dot{h}-[h, \dot{h}]+\frac{[h,[h, \dot{h}]]}{2!}-\frac{[h,[h,[h, \dot{h}]]]}{3!}+\cdots.$$

1. First prove that $d(\exp x, \exp y) \leqslant C_2 |x-y|$:

$$|g^{-1}\dot{g}| \leqslant |\dot{h}| \sum_{k=0}^{\infty} \frac{C^k |h|^k}{k!} = |\dot{h}| e^{C|h|}$$

$$\Rightarrow d(\exp x, \exp y) = \inf_g \int_0^1 |g^{-1}\dot{g}| \, ds \leqslant \inf_h \int_0^1 |\dot{h}| e^{C|h|} \, ds.$$

Take $h(s) = xs + (1-s) y$:

$$d(\exp x, \exp y) \leqslant \int_0^1 |x-y| e^{C(|x| s + (1-s)|y|)} \, ds$$

$$= |x-y| \int_0^1 e^{C(|x|-|y|) s} e^{C|y|} \, ds$$

$$= |x-y| e^{C|y|} \frac{e^{C(|x|-|y|)} - 1}{C(|x|-|y|)}$$

$$= |x-y| \frac{e^{C|x|} - e^{C|y|}}{C(|x|-|y|)} \leqslant |x-y| e^{C \max\{|x|, |y|\}},$$

since $(e^a - e^b)/(a-b) \leqslant e^{\max\{|a|, |b|\}}$. Thus

$$d(\exp x, \exp y) \leqslant |x-y| e^{C \max\{|x|, |y|\}} \leqslant |x-y| e^{C\varepsilon}.$$

2. Now let us prove that $C_1 |x-y| \leqslant d(\exp x, \exp y)$:

$$|g^{-1}\dot{g}| = \left| \sum_{k=0}^{\infty} \frac{(-adh)^k (\dot{h})}{(k+1)!} \right| \geqslant |\dot{h}| \left(1 - \sum_{k=1}^{\infty} \frac{|h|^k C^k}{k!} \right)$$

$$= |\dot{h}| \left(1 - \sum_{k=0}^{\infty} \frac{|h|^k C^k}{k!} + 1 \right) = |\dot{h}| (2 - e^{C|h|}).$$

Consider two cases.

2a. Suppose $\max_s |h(s)| < 2\varepsilon$. Then

$$\int_0^1 |g^{-1}\dot{g}| \, ds \geqslant \int_0^1 |\dot{h}| (2 - e^{C|h|}) \, ds \geqslant (2 - e^{2\varepsilon C}) \int_0^1 |\dot{h}| \, ds$$

$$\geqslant (2 - e^{2\varepsilon C}) \left| \int_0^1 \dot{h} \, ds \right| = (2 - e^{2\varepsilon C}) |x-y|.$$

2b. Take a path h such that $\max_s |h(s)| \geqslant 2\varepsilon$. Denote $t_1 = \min\{t: |h(t)| = \varepsilon\}$, $t_2: \min\{t: |h(t)| = 2\varepsilon\}$. Note that $t_1 < t_2$ since $h(0) = x$ and $|x| < \varepsilon$. Then

$$\int_0^1 |g^{-1}\dot{g}|\, ds \geqslant \int_{t_1}^{t_2} |g^{-1}\dot{g}|\, ds \geqslant \int_{t_1}^{t_2} |\dot{h}|\, (2 - e^{C\,|h|})\, ds$$

$$\geqslant (2 - e^{2\varepsilon C}) \int_{t_1}^{t_2} |\dot{h}|\, ds \geqslant (2 - e^{2\varepsilon C})\, |h(t_2) - h(t_1)|$$

$$\geqslant \varepsilon(2 - e^{2\varepsilon C}) \geqslant \frac{|x - y|}{2}\,(2 - e^{2\varepsilon C}).$$

2a and 2b imply that

$$d(\exp x, \exp y) = \inf_g \int_0^1 |g^{-1}\dot{g}|\, ds \geqslant \frac{|x - y|}{2}\,(2 - e^{2\varepsilon C}). \quad \blacksquare$$

Lemma 7.5. *Take $g(t)$: $[0, 1] \to G_{\mathrm{CM}}$, $g \in C^1_{\mathrm{CM}}$. Suppose that $g(0) = e$, $\|g(t) - e\| < 1$ for any $t \in [0, 1]$ and $|\log g(1)| < \varepsilon$, where ε is the same as in Lemma 7.4. Then $g(1) \in G_\infty$.*

Proof. Let $h = \log g = \sum_{n=1}^{\infty} ((-1)^{n-1}/n)(g(t) - I)^n$. Then by Lemma 7.3 h is a unique solution in $\mathfrak{g}_\infty$ of the ordinary differential Eq. 7.1. In particular, it implies that $g(t) \in \mathfrak{g}_\infty$. Therefore, there are $H_n \in \mathfrak{g}_n$ (projections of $h(1)$ onto $\mathfrak{g}_n$) such that $|h(1) - H_n| \xrightarrow[n \to \infty]{} 0$.

By Lemma 7.4

$$d(e^{h(1)}, e^{H_n}) \leqslant C_2\, |h(1) - H_n|.$$

A similar estimate holds for d_n instead of d. By a direct calculation $e^{h(t)} = g(t)$. Thus $g(1) \in G_\infty$, since $e^{H_n} \in G_n$. $\blacksquare$

Proof of Theorem 7.2. 1. $G_\infty \subseteq G_{\mathrm{CM}}$ since $G_N \subseteq G_{\mathrm{CM}}$ for all n and G_∞ is closed in the metric d_∞.

Now we will show that $G_{\mathrm{CM}} \subseteq G_\infty$.

Take $A \in G_\infty$, $B \in G_{\mathrm{CM}}$, a path $g = [0, 1] \to G_{\mathrm{CM}}$, $g \in C^1_{\mathrm{CM}}$, $g(0) = A$, $g(1) = B$. We want to show that $B \in G_\infty$.

Lemma 7.5 proves this in a neighborhood of I. Prove it for a neighborhood of any element of G_∞. Suppose $\|g(t) - A\| < 1$, $d(\log B, \log A) < \varepsilon$.

For any $\delta > 0$ there is $C_\delta \in \bigcup_n G_n$ such that $d(A, C_\delta) < \delta$. Thus

$$d(B, C_\delta) \leqslant d(B, A) + d(A, C_\delta) < \varepsilon + \delta.$$

Therefore $d(C_\delta^{-1}B, e) < \varepsilon + \delta$, so by part 1 $C_\delta^{-1}B \in G_\infty$. This means that there is $D_\delta \in \bigcup_n G_n$ such that

$$d(C_\delta^{-1}B, D_\delta) < \delta.$$

Now $d(C_\delta^{-1}B, D_\delta) = d(B, C_\delta D_\delta) < \delta$, $C_\delta D_\delta \in \bigcup_n G_n$. This proves that $B \in G_\infty$.

Now take any A, B, and a path g joining them. Divide $\log g$ into subpaths satisfying the conditions in the first part of the proof.

2. By Lemma 7.3 and Lemma 7.4 exp and log are well defined and differentiable in neighborhoods of the identity and zero respectively. ∎

8. HOLOMORPHIC POLYNOMIALS AND SKELETONS

DEFINITION 8.1. A function $p: I + HS \to \mathbb{C}$ is called *a holomorphic polynomial*, if p is a complex linear combination of finite products of monomials $p_k^{m,l}(X) = (\langle Xf_m, f_l\rangle)^k$, where $\{f_m\}_{m=1}^\infty$ is an orthonormal basis of H. We will denote the space of all such polynomials by $\mathscr{HP}$.

These polynomials are holomorphic because the kth derivative $(D_X^k p)(\beta)$ exists and is complex linear for any $p \in \mathscr{P}$, $X \in I + HS$, $\beta \in \mathfrak{g}_\infty^{\otimes k}$. Indeed, let ξ be any element of $\mathfrak{g}_\infty$, then the derivative of $p_k^{m,l}$ in the direction of ξ is $(Dp_k^{m,l})(X)(\xi) = kp_1^{m,l}(X\xi)\, p_{k-1}^{m,l}(X)$. From this formula we can see that $Dp_k^{m,l}(X)(\xi)$ is complex linear in ξ.

Remark 8.2. Any polynomial $p \in \mathscr{HP}$ can be written in the form $p(X) = \sum_{k=1}^m \prod_{l=1}^{n_m} \operatorname{Tr}(A_{kl}X)$ for some $A_{kl} \in HS$. The converse is not true in general, but the closure in $L^2(I + HS, \mu_t)$ of all functions of the form $\sum_{k=1}^m \prod_{l=1}^{n_m} \operatorname{Tr}(A_{kl}X)$ coincides with the closure of holomorphic polynomials. Therefore the next definition is basis-independent, though the definition of $\mathscr{HP}$ depends on the choice of $\{f_k\}_{k=1}^\infty$.

DEFINITION 8.3. The closure of all holomorphic polynomials in $L^2(I + HS, \mu_t)$ is called $\mathscr{H}L^2(I + HS, \mu_t)$.

LEMMA 8.4 (B. Driver's formula). *Let g be a smooth path in G_{CM} such that $g(0) = I$ and $g \in C^1_{\mathrm{CM}}$. Then for any $p \in \mathscr{HP}$*

$$p(g(s)) = \sum_{n=0}^\infty \int_{\Delta_n(s)} (D^n p)(I)(c(s_1) \otimes \cdots \otimes c(s_n))\, d\mathbf{s},$$

where

$$\Delta_n(s) = \{(s_1, ..., s_n) \in \mathbb{R}^n : 0 \leqslant s_1 \leqslant s_2 \cdots \leqslant s_n \leqslant s\},$$

$$c(s) = g^{-1}(s) \frac{dg}{ds}.$$

Proof. Take a monomial $p(x) = (\langle Xf_m, f_l \rangle)^k$, $\deg p = k$. Similarly to Lemma 5.2 of [4].

$$p(g(s)) = \sum_{n=0}^{N-1} \int_{\Delta_n(s)} (D^n p)(I)(c(s_1) \otimes \cdots \otimes c(s_n))\, d\mathbf{s}$$

$$+ \int_{\Delta_N(s)} (D^N p)(g(s_1))(c(s_1) \otimes \cdots \otimes c(s_N))\, d\mathbf{s}.$$

We want to estimate the remainder

$$\left| \int_{\Delta_N(s)} (D^N p)(g(s_1))(c(s_1) \otimes \cdots \otimes c(s_N))\, d\mathbf{s} \right|$$

$$\leqslant \int_{\Delta_N(s)} |(D^N p)(g(s_1))(c(s_1) \otimes \cdots \otimes c(s_N))|\, d\mathbf{s}.$$

Note that if $k = \deg p \leqslant N$ then

$$(D^N p)(X)(\xi_1 \otimes \cdots \otimes \xi_N)$$

$$= \sum p_1(X\xi^{\alpha_1})\, p_1(X\xi^{\alpha_2}) \cdot \cdots \cdot p_1(X\xi^{\alpha_i}) \cdot \cdots \cdot p_1(X\xi^{\alpha_k}),$$

where $\alpha_i = \langle a_1^i, ..., a_N^i \rangle$, $a_j^i = 0$ or 1, $\xi^{\alpha_i} = \xi_1^{a_1^i} \cdots \xi_N^{a_N^i}$, $\sum_{i=1}^k \alpha_i = \langle 1, ..., 1 \rangle$, $p_1(Y) = \langle Xf_m, f_l \rangle$, and the sum is taken over all possible $\alpha = (\alpha_1, ..., \alpha_k)$.

For $\xi \in HS$, $X \in I + HS$.

$$|p_1(X\xi)| \leqslant (\|(X-I)\, \xi\|_{\mathrm{HS}} + \|\xi\|_{\mathrm{HS}}) \leqslant \|\xi\|_{\mathrm{HS}}\, (\|(X-I)\|_{\mathrm{HS}} + 1).$$

The number of all such α is k^N therefore

$$|(D^n p)(X)(\xi_1 \otimes \cdots \otimes \xi_N)|$$

$$\leqslant \|\xi_1\|_{\mathrm{HS}} \cdots \|\xi_N\|_{\mathrm{HS}}\, (\|(X-I)\|_{\mathrm{HS}} + 1)\, k^N.$$

Denote

$$\|c\|_1 = \int_0^1 |c(t)|\, dt.$$

Thus

$$\int_{\Delta_N(s)} |(D^N p)(g(s_1))(c(s_1)\otimes \cdots \otimes c(s_N))|\, d\mathbf{s}$$

$$\leqslant \|c\|_1^N \sup_{0\leqslant t\leqslant s} (\|(g(t)-I)\|_{\mathrm{HS}}+1)^k\, k^N \int_{\Delta_N(s)} d\mathbf{s}$$

$$= \|c\|_1^N \sup_{0\leqslant t\leqslant s} (\|(g(t)-I)\|_{\mathrm{HS}}+1)^k\, k^N \frac{s^N}{N!} \xrightarrow[n\to\infty]{} 0.$$

Similarly we can prove this formula for any holomorphic polynomial. ▮

THEOREM 8.5. *Suppose*

$$p_n \xrightarrow[n\to\infty]{L^2(I+HS,\mu_t)} f, \qquad p_n \in \mathscr{H}\mathscr{P}.$$

Then there is a holomorphic function $\tilde{f}$, *a skeleton of* f, *on* G_{CM} *such that*

$$p_n(x) \to \tilde{f}(x) \qquad \textit{for any} \quad x \in G_{\mathrm{CM}}.$$

Proof. As we know from Theorem 4.4, the map $(1-D)_I^{-1}$ is an isometry from $\mathscr{H}L^2(I+HS,\mu_t)$ to J_t^0. Thus $(1-D)_I^{-1}\, p_n$ converges in J_t^0 to some $\alpha=\sum \alpha_n$. Define

$$\tilde{f}(x) = \sum_{n=0}^{\infty} \int_{\Delta_n(1)} \alpha_n(c(s_1)\otimes \cdots \otimes c(s_n))\, d\mathbf{s}.$$

Take a path $g=[0,1]\to G_{\mathrm{CM}}$, $g(0)=e$, $g(1)=X$. By Lemma 8.4

$$p_m(X) = \sum_{n=0}^{\infty} \int_{\Delta_n(1)} (D^n p_m)(I)(c(s_1)\otimes \cdots \otimes c(s_n))\, d\mathbf{s},$$

so

$$|p_m(X)-\tilde{f}(x)| \leqslant \sum_{n=0}^{\infty} \int_{\Delta_n(1)} |((D^n p_m)(I)-\alpha_n)(c(s_1)\otimes \cdots \otimes c(s_n))|\, d\mathbf{s}$$

$$\leqslant \sum_{n=0}^{\infty} \int_{\Delta_n(1)} |(D^n p_m)(I)-\alpha_n|\, |c(s_1)\otimes \cdots \otimes c(s_n)|\, d\mathbf{s}$$

$$\leqslant \sum_{n=0}^{\infty} |(D^n p_m)(I)-\alpha_n| \int_{\Delta_n(1)} |c(s_1)\otimes \cdots \otimes c(s_n)|\, d\mathbf{s}.$$

Thus

$$
\begin{aligned}
|p_m(X)-\tilde{f}(x)| &\leqslant \sum_{n=0}^{\infty} |(D^n p_m)(I)-\alpha_n| \frac{\|c\|_1^n}{n!} \\
&= \sum_{n=0}^{\infty} \frac{|(D^n p_m)(I)-\alpha_n|\, t^{n+2}}{\sqrt{n!}} \frac{\|c\|_1^n}{\sqrt{n!}\, t^{n/2}} \\
&\leqslant \left(\sum_{n=0}^{\infty} \frac{|(D^n p_m)(I)-\alpha_n|^2\, t^n}{n!}\right)\left(\sum_{n=0}^{\infty} \frac{\|c\|_1^{2n}}{n!\, t^n}\right) \\
&= \|(Dp_m)(I)-\alpha\|_t^2\, e^{\|c\|_1^2/t} \xrightarrow[m\to\infty]{} 0.
\end{aligned}
$$

Now we will show that $\tilde{f}$ is holomorphic. Take $\xi \in \mathfrak{g}$, $g \in C^1_{\mathrm{CM}}$, $g(0)=e$, $g(1)=X$. Define two paths in G_{CM}

$$
g_1(s)=\begin{cases} g(s), & 0\leqslant s\leqslant 1 \\ g(1), & 1\leqslant s\leqslant 1+t \end{cases}
$$

$$
g_2(s)=\begin{cases} g(s), & 0\leqslant s\leqslant 1 \\ g(1)\, e^{(s-1)\,\xi}, & 1\leqslant s\leqslant 1+t. \end{cases}
$$

Let $c(s)=g^{-1}(s)\, dg/ds$; then

$$
c_1(s)=g_1^{-1}(s)\frac{dg_1}{ds}=\begin{cases} c(s), & 0\leqslant s\leqslant 1 \\ 0, & 1\leqslant s\leqslant 1+t \end{cases}
$$

$$
c_2(s)=g_2^{-1}(s)\frac{dg_2}{ds}=\begin{cases} c(s), & 0\leqslant s\leqslant 1 \\ \xi, & 1\leqslant s\leqslant 1+t. \end{cases}
$$

Then by the definition of $\tilde{f}$

$$
\begin{aligned}
&\tilde{f}(Xe^{t\xi})-\tilde{f}(X) \\
&\quad= \sum_{n=0}^{\infty}\int_{\Delta_n(1+t)} \alpha_n(c_1(s_1)\otimes\cdots\otimes c_1(s_n)-c_2(s_1)\otimes\cdots\otimes c_2(s_n))\, d\mathbf{s}_n \\
&\quad= \sum_{n=0}^{\infty}\int_1^{1+t}\int_{\Delta_{n-1}(s_n)} \alpha_n(c_1(s_1)\otimes\cdots\otimes c_1(s_n) \\
&\qquad - c_2(s_1)\otimes\cdots\otimes c_2(s_n))\, d\mathbf{s}_{n-1}\, ds_n
\end{aligned}
$$

$$
\begin{aligned}
&= \sum_{n=0}^{\infty} \int_{\Delta_{n-1}(1)} \alpha_n(c(s_1) \otimes \cdots \otimes c(s_{n-1}) \otimes t\xi)\, d\mathbf{s}_{n-1} \\
&\quad + \sum_{n=0}^{\infty} \int_1^{1+t} \int_1^{s_n} \int_{\Delta_{n-2}(s_{n-1})} \alpha_n(c_1(s_1) \otimes \cdots \otimes c_1(s_n) \\
&\quad - c_2(s_1) \otimes \cdots \otimes c_2(s_n))\, d\mathbf{s}_{n-2}\, ds_{n-1}\, ds_n \\
&= \sum_{n=0}^{\infty} t \int_{\Delta_{n-1}(1)} \alpha_n(c(s_1) \otimes \cdots \otimes c(s_{n-1}) \otimes \xi)\, d\mathbf{s}_{n-1} \\
&\quad + \sum_{n=0}^{\infty} \int_1^{1+t} \int_1^{s_n} \int_{\Delta_{n-2}(s_{n-1})} \alpha_n(c(s_1) \otimes \cdots \otimes c(s_{n-2}) \otimes \xi \otimes \xi) \\
&\quad \times d\mathbf{s}_{n-2}\, ds_{n-1}\, ds_n.
\end{aligned}
$$

Thus

$$
\begin{aligned}
&\left| \frac{\tilde{f}(Xe^{t\xi}) - \tilde{f}(X)}{t} - \int_{\Delta_{n-1}(1)} \alpha_n(c(s_1) \otimes \cdots \otimes c(s_{n-1}) \otimes \xi)\, d\mathbf{s}_{n-1} \right| \\
&\leqslant \frac{1}{t} \int_1^{1+t} (s_n - 1) \int_{\Delta_{n-2}(s_{n-1})} \alpha_n(c(s_1) \otimes \cdots \otimes c(s_{n-2}) \otimes \xi \otimes \xi) \\
&\quad \times d\mathbf{s}_{n-2}\, ds_{n-1}\, ds_n \xrightarrow[n \to 0]{} 0
\end{aligned}
$$

since $0 \leqslant s_n - 1 \leqslant t$.

This means that the derivative of $\tilde{f}$ at X in the direction ξ is

$$
(\xi \tilde{f})(X) = \int_{\Delta_{n-1}(1)} \alpha_n(c(s_1) \otimes \cdots \otimes c(s_{n-1}) \otimes \xi)\, d\mathbf{s}_{n-1}
$$

and it is complex linear. ∎

Remark 8.6. The convergence is uniform on bounded sets. In fact, one can show that the derivatives also converge uniformly on bounded sets.

THEOREM 8.7. 1. *$\mathcal{HP} \subset L^p(I + HS, \mu_t)$, for any $p > 1$.*

2. *$\|f\|_{t,n} \xrightarrow[n \to \infty]{} \|f\|_t$ for any $f \in \mathcal{HP}$.*

3. *The embedding of $\mathcal{HP}$ into $\mathcal{H}^t(G_\infty)$ can be extended to an isometry from $\mathcal{H}L^2(I + HS, \mu_t)$ into $\mathcal{H}^t(G_\infty)$.*

Proof. 1.

$$\begin{aligned}\int_{I+HS} |p_k^{m,l}(X)|^p \, \mu_t(dX) &= E\,|p_k^{m,l}(Y_t+I)|^p \\ &= E\,|(Y_t)_{ml}+\delta_{ml}|^{pk} \\ &\leqslant E 2^{pk-1}(|(Y_t)_{ml}|^{pk}+1) \\ &\leqslant 2^{pk-1} E(\|Y_t\|_{\mathrm{HS}}^{pk}+1)<\infty\end{aligned}$$

by Lemma 5.7. Thus $\mathscr{H}\mathscr{P} \subset L^p(I+HS, \mu_t)$ for any $p>1$.

2. From the estimate on $E\,\|Y_t\|_{\mathrm{HS}}^p$ in Proposition 5.6 we can find $C(p,t)$ such that $\|\,|p|^2\|_{t,n(k)} \leqslant C(p,t)$ for any k. Then apply Lemma 5.7 to $f=|p|^2$.

3. By part 1 of Theorem 8.7 $\mathscr{H}\mathscr{P} \subset L^2(I+HS, \mu_t)$. In addition, by part 2 of Theorem 8.7

$$\|p\|_{t,n} \xrightarrow[n\to\infty]{} \|p\|_t, \qquad p\in\mathscr{H}\mathscr{P}.$$

Therefore $\|p\|_{t,\infty}=\|p\|_t$ and so the embedding is an isometry.

The space $\mathscr{H}L^2(I+HS, \mu_t)$ is the closure of $\mathscr{H}\mathscr{P}$, therefore the isometry extends to it from $\mathscr{H}\mathscr{P}$. ∎

Remark 8.8. If f is an element of $\mathscr{H}L^2(I+HS, \mu_t)$, g is its image under the isometry in this theorem and $\tilde{f}$ is the skeleton of f, then $\tilde{f}|_{G_\infty}=g$.

COROLLARY 8.9. *The space $\mathscr{H}^t(G_\infty)$ is an infinite dimensional Hilbert space.*

9. EXAMPLES

Denote $HS_{n\times n}=\{A\colon \langle Af_m, f_k\rangle=0 \text{ if } \max(m,k)>n\}$. Take a basis $\{e_k\}_{k=1}^{\infty}$ of HS such that $\{e_k\}_{k=1}^{2n^2}$ is a basis of $HS_{n\times n}$. By B^{T} we will denote the transpose of the operator B, i.e., $B^{\mathrm{T}}=(\operatorname{Re} B)^*+i(\operatorname{Im} B)^*$.

EXAMPLE 9.1. We begin with the definition of the Hilbert–Schmidt complex orthogonal group.

DEFINITION 9.1. *The Hilbert–Schmidt complex orthogonal group SO_{HS} is the connected component containing the identity I of the group $O_{\mathrm{HS}}=\{B\colon B-I\in HS,\ B^{\mathrm{T}}B=BB^{\mathrm{T}}=I\}$. The Lie algebra of skew-symmetric Hilbert–Schmidt operators will be denoted by $\mathfrak{so}_{\mathrm{HS}}=\{A\colon A\in HS,\ A^{\mathrm{T}}=-A\}$.*

Let Q be a symmetric positive trace class operator on $\mathfrak{so}_{\mathrm{HS}}$ and let the inner product be defined by $\langle A, B\rangle = \langle Q^{-1/2}A, Q^{-1/2}B\rangle_{\mathrm{HS}}$. In [7] we showed that if Q is the identity operator, then all Hilbert spaces we consider are isomorphic to $\mathbb{C}$; that is, there are no nonconstant holomorphic functions. As in Section 5 we identify Q with its extension by 0 to the orthogonal complement of $\mathfrak{so}_{\mathrm{HS}}$.

Define groups $G_n = SO(n, \mathbb{C}) = \{B \in SO_{\mathrm{HS}}, B - I \in HS_{n\times n}\}$. These groups are isomorphic to the special complex orthogonal groups of $\mathbb{C}^n$. Their Lie algebras are Lie $(SO(n, \mathbb{C})) = \mathfrak{so}(n, \mathbb{C}) = \{A \in HS_{n\times n}, A^{\mathrm{T}} = -A\}$ with an inner product $\langle A, B\rangle_n = \langle (P_n Q P_n)^{-1/2} A, (P_n Q P_n)^{-1/2} B\rangle_{\mathrm{HS}}$. Here $P_n = P_{\mathfrak{so}(n, \mathbb{C})}$. We assume that all $\mathfrak{so}(n, \mathbb{C})$ are invariant subspaces of Q. The groups $SO(n, \mathbb{C})$ are not simply connected, therefore we have isometries from $\mathscr{H}^t(SO_\infty)$ and $\mathscr{H}L^2(I + HS, \mu_t)$ to J_t^0, but not an isomorphism between $\mathscr{H}^t(SO_\infty)$ and J_t^0. In addition to the properties of the heat kernel measure described in this paper, we showed in [7] that the process $Y_t + I$ actually lives in the group SO_{HS}.

Example 9.2. The Hilbert–Schmidt complex symplectic group is defined similarly to the Hilbert–Schmidt complex orthogonal group.

Definition 9.2. *The Hilbert–Schmidt complex symplectic group* Sp_{HS} is the group of operators $X = \left(\begin{smallmatrix} A & B \\ C & D\end{smallmatrix}\right)$ such that $A - I$, $D - I$, B, $C \in HS$, and $X^{\mathrm{T}} J X = J$, where $J = \left(\begin{smallmatrix} 0 & -I \\ I & 0\end{smallmatrix}\right)$. The Lie algebra is $\mathfrak{sp}_{\mathrm{HS}} = \{X = \left(\begin{smallmatrix} A & B \\ C & C\end{smallmatrix}\right) : A, B, C, D \in HS, X^{\mathrm{T}} J + JX = 0\}$.

The corresponding finite dimensional groups are isomorphic to the classical symplectic complex groups $\mathrm{Sp}(n, \mathbb{C}) = \{X = \left(\begin{smallmatrix} A & B \\ C & D\end{smallmatrix}\right) \in \mathrm{Sp}_{\mathrm{HS}}, A - I, D - I, B, C \in HS_{n\times n}\}$ with Lie algebras $\mathfrak{sp}(n, \mathbb{C}) = \{X = \left(\begin{smallmatrix} A & B \\ C & D\end{smallmatrix}\right) \in \mathfrak{sp}_{\mathrm{HS}} : A, B, C, D \in HS_{n\times n}\}$. An inner product on $\mathfrak{sp}_{\mathrm{HS}}$ and $\mathfrak{sp}(n, \mathbb{C})$ is defined in the same way as in Example 9.1. Similarly to $\mathfrak{so}_{\mathrm{HS}}$ if Q is the identity operator, the corresponding J_0^t is trivial by Theorem 4.6. The groups $\mathrm{Sp}(n, \mathbb{C})$ are simply connected; therefore the isometry from $\mathscr{H}^t(\mathrm{Sp}_\infty)$ to J_t^0 is surjective. Similar to SO_{HS} the process $Y_t + I$ lives in $\mathrm{Sp}_{\mathrm{HS}}$.

Statement 9.3. *$Y_t + I$ lies in Sp_{HS} for any $t > 0$ with probability 1.*

Proof. We need to check that $(Y_t + I) J (Y_t + I)^{\mathrm{T}} = J$ with probability 1 for any $t > 0$. To do this we will apply Itô's formula to $G(Y_t)$, where G is defined as follows: $G(Y) = \Lambda(YJY^{\mathrm{T}} + YJ + JY^{\mathrm{T}})$, Λ is a linear real bounded functional from HS to $\mathbb{R}$.

In order to use Itô's formula we must verify several properties of the process Y_t and the mapping G:

1. $B(Y_s)$ is an L_2^0-valued process stochastically integrable on $[0, T]$.

2. G and the derivatives G_t, G_Y, G_{YY} are uniformly continuous on bounded subsets of $[0, T] \times HS$.

Proof of 1. See 1 in the proof of Theorem 5.1

Proof of 2. Let us calculate G_t, G_Y, G_{YY}. First of all, $G_t = 0$. For any $S \in HS$,

$$G_Y(Y)(S) = \Lambda(SJY^{\mathrm{T}} + YJS^{\mathrm{T}} + SJ + JS^{\mathrm{T}}).$$

For any $S, T \in HS$.

$$G_{YY}(Y)(S \otimes T) = \Lambda(SJT^{\mathrm{T}} + TJS^{\mathrm{T}}).$$

Thus condition 2 is satisfied.

We will use the notation

$$G_Y(Y)(S) = \langle \bar{G}_Y(Y), S \rangle_{\mathrm{HS}},$$
$$G_{YY}(Y)(S \otimes T) = \langle \bar{G}_{YY}(Y)\, S, T \rangle_{\mathrm{HS}},$$

where $\bar{G}_Y$ is an element of HS and $\bar{G}_{YY}$ is an operator on HS corresponding to the functionals $G_Y \in HS^*$ and $G_{YY} \in (HS \otimes HS)^*$.

Now we can apply Itô's formula to $G(Y_t)$:

$$G(Y_t) = \int_0^t \langle \bar{G}_Y(Y_s), B(Y_s)\, dW_s \rangle_{\mathrm{HS}}$$
$$+ \int_0^t \tfrac{1}{2} \operatorname{Tr}[G_{YY}(Y_s)(B(Y_s)\, Q^{1/2})(B(Y_s)\, Q^{1/2})^*]\, ds. \tag{9.1}$$

Let us calculate the two integrands in (9.1) separately.

The first integrand is

$$\begin{aligned}
&\langle \bar{G}_Y(Y_s), B(Y_s)\, dW_s \rangle_{\mathrm{HS}} \\
&\quad = \langle \bar{G}_Y(Y_s), (Y_s + I)\, dW_s \rangle_{\mathrm{HS}} \\
&\quad = \Lambda(((Y_s + I)\, dW_s)\, JY_s^{\mathrm{T}} + Y_s J((Y_s + I)\, dW_s)^{\mathrm{T}} \\
&\qquad + ((Y_s + I)\, dW_s)\, J + J((Y_s + I)\, dW_s)^{\mathrm{T}}) \\
&\quad = \Lambda((Y_s + I)\, dW_s\, JY_s^{\mathrm{T}} + Y_s J(dW_s^{\mathrm{T}}(Y_s^{\mathrm{T}} + I)) \\
&\qquad + (Y_s + I)\, dW_s\, J + J\, dW_s^{\mathrm{T}}(Y_s^{\mathrm{T}} + I)) \\
&\quad = \Lambda((Y_s + I)\, dW_s\, JY_s^{\mathrm{T}} - Y_s\, dW_s\, J(Y_s^{\mathrm{T}} + I) \\
&\qquad + (Y_s + I)\, dW_s\, J - dW_s\, J(Y_s^{\mathrm{T}} + I)) = 0,
\end{aligned}$$

since W_t is a $\mathfrak{sp}_{\mathrm{HS}}$-valued process.

The second integrand is

$$
\begin{aligned}
&\tfrac{1}{2}\operatorname{Tr}[G_{YY}(Y_s)(B(Y_s)\,Q^{1/2})(B(Y_s)\,Q^{1/2})^*] \\
&\quad=\tfrac{1}{2}\sum_{n=1}^{\infty}\langle \bar{G}_{YY}(Y_s)\,B(Y_s)\,Q^{1/2}e_n, B(Y_s)\,Q^{1/2}e_n\rangle_{\mathrm{HS}} \\
&\quad=\tfrac{1}{2}\sum_{n=1}^{\infty}\Lambda(2((Y_s+I)\,\xi_n)\,J((Y_s+I)\,\xi_n)^{\mathrm{T}}) \\
&\quad=\sum_{n=1}^{\infty}\Lambda((Y_s+I)\,\xi_n J\xi_n^{\mathrm{T}}(Y_s^{\mathrm{T}}+I)) \\
&\quad=-\sum_{n=1}^{\infty}\Lambda((Y_s+I)\,\xi_n^2 J(Y_s^{\mathrm{T}}+I))=0
\end{aligned}
$$

by Lemma 5.2. This shows that the stochastic differential of G is zero, so $G(Y_t)=0$ for any $t>0$. ∎

Example 9.3. Let G be a group of diagonal (infinite) complex matrices $\operatorname{diag}(1+a_1, ..., 1+a_i, ...)$, where $\sum_{i=1}^{\infty}|a_i|^2<\infty$, $a_i\neq -1$. Then $G\cong \prod_{i=1}^{\infty}\mathbb{C}\setminus\{0\}$. This group is abelian and is not simply connected. The Lie algebra $\mathfrak{g}$ is an algebra of diagonal matrices $\operatorname{diag}(a_1, ..., a_i, ...)$, where $\sum_{i=1}^{\infty}|a_i|^2<\infty$. Note that if we take operator Q as before to define a new inner product on $\mathfrak{g}$, this inner product is $\mathrm{Ad}_{G_{\mathbb{R}}}$-invariant. The process is a sum of processes in $\mathbb{C}$,

$$dY_t^i=B^i(Y_t^i)\,dW_t^i.$$

We can also write this equation in terms of the real and imaginary parts of Y_t^i,

$$
\begin{aligned}
dY_t^{i,1}&=(Y_t^{i,1}+1)\,dW_t^{i,1}-Y_t^{i,2}\,dW_t^{i,2}\\
dY_t^{i,2}&=Y_t^{i,2}\,dW_t^{i,1}+(Y_t^{i,2}+1)\,dW_t^{i,2},
\end{aligned}
$$

where $Y_t^{i,1}=\operatorname{Re}Y_t^i$ and $Y_t^{i,2}=\operatorname{Im}Y_t^i$.

Example 9.4. The following example shows that there exist Lie algebras with non trivial J_0^t for which the condition on the Lie bracket in Theorem 7.2 is satisfied. Let $\mathfrak{g}$ be equal to $Q^{1/2}HS$, where the operator Q is defined as follows. We will view the elements of HS as infinite matrices. Denote by e_{ij} an infinite matrix whose entries are all zero except the one equal to 1 at the intersection of the ith row and jth column. These matrices form an orthonormal basis of HS. We assume that Q is diagonal in this basis, namely, $Qe_{ij}=e^{-(i+j)}e_{ij}$. Note that this Q is a positive trace class operator, so we can construct the corresponding heat kernel measure. Thus the space $\mathscr{H}^t(G_\infty)$ contains all holomorphic polynomials and therefore the space J_0^t is not trivial. Now let us verify that the condition on the Lie bracket is satisfied. For any x and y in $\mathfrak{g}$

$$\begin{aligned}
|xy|^2 &= \sum_{i,m} e^{i+m} \Big(\sum_j x_{i,j} y_{j,m} \Big)^2 \\
&= \sum_{i,m} \Big(\sum_j e^{-j} e^{(1/2)(i+j)} x_{i,j} e^{(1/2)(j+m)} y_{j,m} \Big)^2 \\
&\leqslant \sum_{i,m} \Big(\sum_j e^{(1/2)(i+j)} x_{i,j} e^{(1/2)(j+m)} y_{j,m} \Big)^2 \\
&\leqslant \sum_{i,m} \Big(\sum_j e^{i+j} x_{i,j}^2 \Big) \Big(\sum_k e^{k+m} y_{k,m}^2 \Big) \\
&= \sum_{i,j} e^{i+j} x_{i,j}^2 \sum_{m,k} e^{k+m} x_{m,k}^2 = |x|^2\, |y|^2.
\end{aligned}$$

Thus $|[x,y]| \leqslant 2\,|x|\,|y|$. A similar construction can be done for the algebras considered in Example 9.1 and Example 9.2.

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